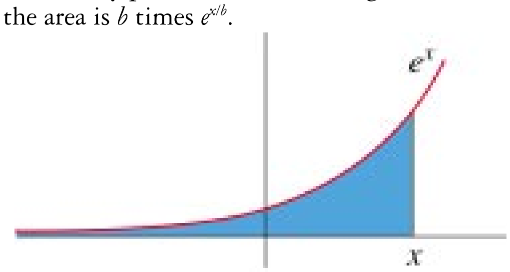
You might think you need calculus to determine the area between the tire tracks made by this bike, ridden by Jason McIlhaney, BS 2000. Surprisingly, geometry offers another way of solving it—without formulas.

A Visual Approach to Calculus Problems

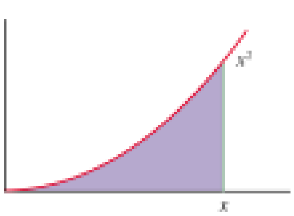
by Tom M. Apostol

Calculus is a beautiful subject with a host of dazzling applications. As a teacher of calculus for more than 50 years and as an author of a couple of textbooks on the subject, I was stunned to learn that many standard problems in calculus can be easily solved by an innovative visual approach that makes no use of formulas. Here's a sample of three such problems:

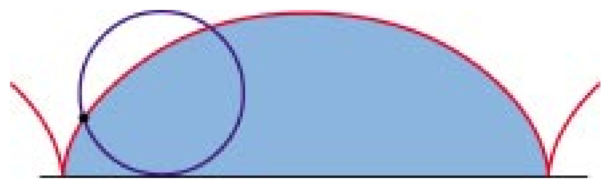
Problem 1. Find the area of the region under an exponential curve. In the graph of the exponential function $y = e^x$, below, we want the area of the blue region between the curve and the x-axis and along the interval from minus infinity up to any point x. Integral calculus reveals that the answer is e^x. And if the equation of the curve is $y = e^{x/b}$, where b is any positive constant, integration tells us the area is b times $e^{x/b}$.



Problem 2. Find the area of a parabolic segment (left)—the purple region below the graph of the parabola $y = x^2$ from 0 to x. The area of the parabolic segment was first calculated by Archimedes more than 2000 years ago by a method that laid the foundations for integral calculus. Today, every freshman calculus student can solve this problem: Integration of x^2 gives $x^3/3$.



Problem 3. Find the area of the region under one arch of a cycloid (next column). A cycloid is the path traced out by a fixed point on the boundary of a circular disk that rolls along a horizontal line, and we want the area of the region shown in blue. This problem can also be done by calculus but it is more difficult than the first two. First, you have to find an equation for the cycloid, which is not exactly trivial. Then you have to integrate this to get the required area. The answer is three times the area of the rolling circular disk.

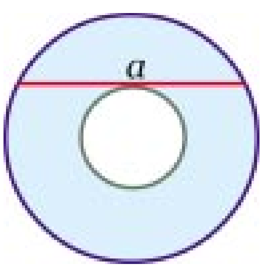


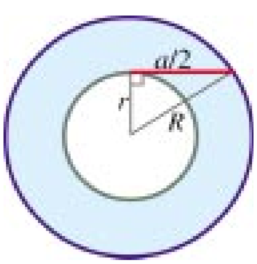
These classic problems can also be solved by a new method that relies on geometric intuition and is easily understood even by very young students. You don't need any equations. Moreover, the new method also solves some problems that can't be done *with* calculus.

The method was conceived in 1959 by Mamikon A. Mnatsakanian, then an undergraduate at Yerevan University in Armenia. When he showed his method to Soviet mathematicians they dismissed it out of hand and said, "It can't be right—you can't solve calculus problems that easily." He went on to get a PhD in physics, was appointed a professor of astrophysics at the University of Yerevan, and became an international expert in radiative transfer theory. He also continued to develop his powerful geometric methods. He eventually published a paper outlining them in 1981, but it seems to have escaped notice, probably because it appeared in Russian in an Armenian journal with limited circulation (*Proceedings of the Armenian Academy of Sciences*, vol. 73, no. 2, pages 97–102).

Mamikon came to California about a decade ago to work on an earthquake-preparedness program for Armenia, and when the Soviet government collapsed, he was stranded in the United States without a visa. With the help of a few mathematicians in Sacramento and at UC Davis, he was granted status as an "alien of extraordinary ability." While working for the California Department of Education and at UC Davis, he further developed his methods into a universal teaching tool using hands-on and computer activities, as well as pictures. He has taught these methods at UC Davis and in Northern California classrooms, ranging from Montessori elementary schools to inner-city public high schools, and he has demonstrated them at teacher conferences. Students and teachers alike have responded enthusiastically, because the methods are vivid and dynamic and don't require the algebraic formalism of trigonometry or calculus.

About four years ago, Mamikon showed up at *Project MATHEMATICS!* headquarters and convinced me that his methods have the potential to make a significant impact on mathematics education, especially if they are combined with visualization tools of modern technology. Since then we have published several joint papers on innovative ideas in elementary mathematics.

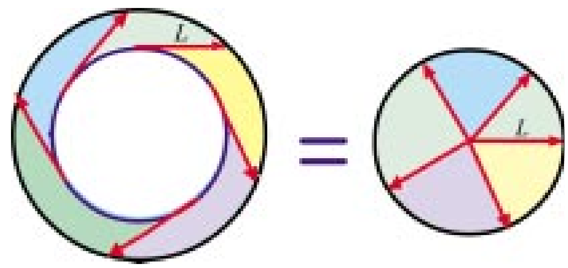




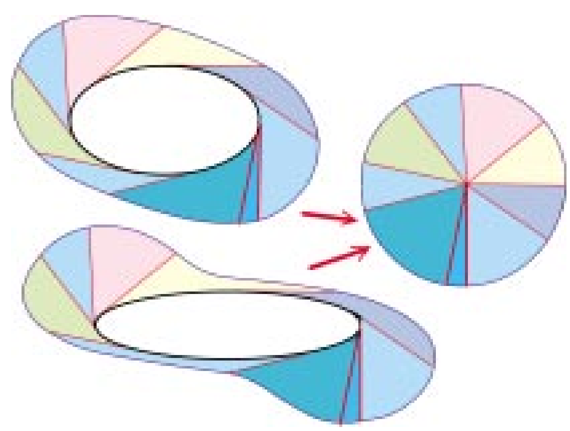
Like all great discoveries, the method is based on a simple idea. It started when young Mamikon was presented with the classical geometry problem, involving two concentric circles with a chord of the outer circle tangent to the inner one, illustrated at left. The chord has length *a,* and the problem is: Find the area of the ring between the circles. As the late Paul Erdös would have said, any baby can solve this problem. Now look at the diagram below it. If the inner circle has radius r its area is πr^2, and if the outer circle has radius R, its area is πR^2, so the area of the ring is equal to $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$. But the two radii and the tangent form a right triangle with legs r and $a/2$ and hypotenuse R, so by the Pythagorean Theorem, $R^2 - r^2 = (a/2)^2$, hence the ring has the area $\pi a^2/4$. Note that the final answer depends only on a and not on the radii of the two circles.

If we knew in advance that the answer depends only on a, we could find it another way: Shrink the inner circle to a point, and the ring collapses to a disk of diameter a, with an area equal to $\pi a^2/4$.

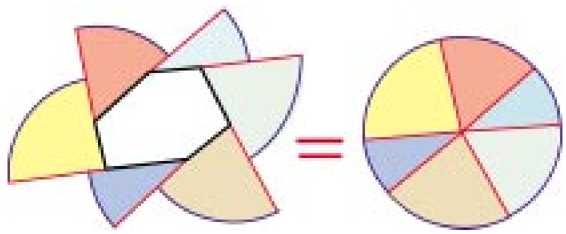
Mamikon wondered if there was a way to see why the answer depends only on the length of the chord. Then he thought of formulating the problem in a dynamic way. Take half the chord and think of it as a vector of length L tangent to the inner circle. By moving this tangent vector around the inner circle, we see that it sweeps out the ring between the two circles. (But it's obvious that the area is being swept due to pure rotation.) Now, translate each tangent vector parallel to itself so that the point of tangency is brought to a common point. As the tangent vector moves around the inner circle, the translated vector rotates once around this common point and traces out a circular disk of radius L. So the tangent vectors sweep out a circular disk as though they were all centered at the same point, as illustrated below. And this disk has the same area as the ring.



Mamikon realized that this dynamic approach would also work if the inner circle was replaced by an arbitrary oval curve. Below you can see the same idea applied to two different ellipses. As the tangent segment of constant length moves once around each ellipse, it sweeps out a more general annular shape that we call an oval ring.



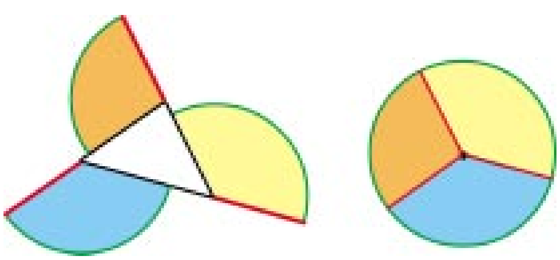
Left: Students at the Southland Park Montessori Elementary School in Sacramento play with magnetic manipulative wedges, shaping them into a circular disk. Pushing with your finger in the center can turn it into an oval ring of any shape.



Again, we can translate each tangent segment parallel to itself so that the point of tangency is brought to a common point. As the tangent moves around the oval, the translated segments trace out a circular disk whose radius is that constant length. So, the area of the oval ring should be the area of the circular disk.

The Pythagorean Theorem can't help you find the areas for these oval rings. If the inner oval is an ellipse, you can calculate the areas by integral calculus (which is not a trivial task); if you do so, you'll find that all of these oval rings have equal areas depending only on the length of the tangent segment.

Is it possible that the same is true for any convex simple closed curve? The diagram below illustrates the idea for a triangle.

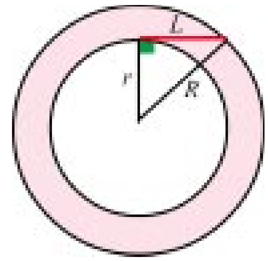


As the tangent segment moves along an edge, it doesn't change direction so it doesn't sweep out any area. As it moves around a vertex from one edge to the next, it sweeps out part of a circular sector. And as it goes around the entire triangle, it sweeps out three circular sectors that, together, fill out a circular disk, as shown to the right.

The same is true for any convex polygon, as illustrated above.

The area of the region swept out by a tangent segment of given length moving around any convex polygon is equal to the area of a circular disk whose radius is that length. Therefore the same is true for any convex curve that is a limit of convex polygons. This leads us to

Mamikon's Theorem for Oval Rings: *All oval rings swept out by a line segment of given length with one endpoint tangent to a smooth closed plane curve have equal areas, regardless of the size or shape of the inner curve. Moreover, the area depends only on the length L of the tangent segment and is equal to* πL^2*, the area of a disk of radius L, as if the tangent segment was rotated about its endpoint.*

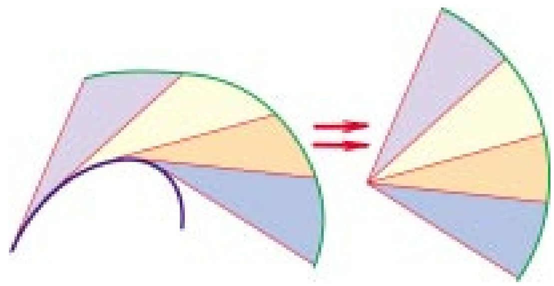


Incidentally, Mamikon's theorem for oval rings provides a new proof of the Pythagorean Theorem, as illustrated at right.

If the inner curve is a circle of radius r, the outer curve will also be a circle (of radius R, say), so the area of the oval ring will be equal to the difference $\pi R^2 - \pi r^2$. But by Mamikon's theorem, the area of the oval ring is also equal to πL^2, where L is the constant length of the tangent segments. By equating areas we find $R^2 - r^2 = L^2$, from which we get $R^2 = r^2 + L^2$, the Pythagorean Theorem.

Now we can illustrate a generalized version of Mamikon's theorem. The lower curve in the diagram at the top of the next page is a more or less arbitrary smooth curve. The set of all tangent segments of constant length defines a region that is bounded by the lower curve and an upper curve traced out by the segment's other extremity. The

Right: Mamikon helps children at the Montessori School trace a tractrix with a bicycle.



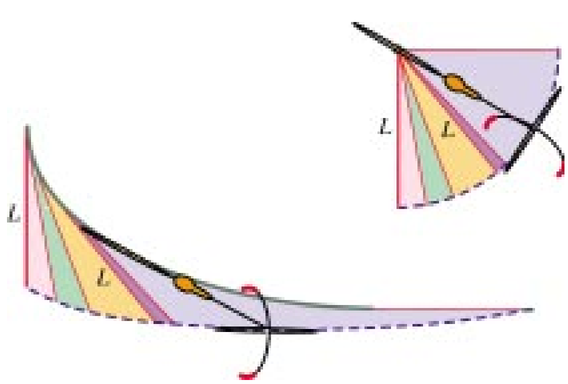
exact shape of this region will depend on the lower curve and on the length of the tangent segments. We refer to this region as a *tangent sweep*.

When each segment is translated to bring the points of tangency together as before, as shown in the right-hand diagram above, the set of translated segments is called the *tangent cluster*. When the tangent segments have constant length, as in this figure, the tangent cluster is a circular sector whose radius is that constant length.

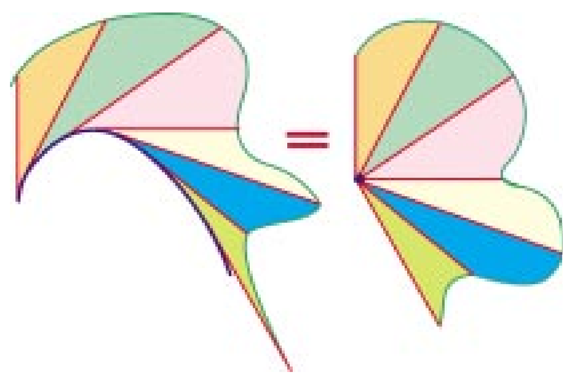
By the way, we could also translate the segments so that the *other* endpoints are brought to a common point. The resulting tangent cluster would be a symmetric version of the cluster in the right-hand figure. Now we can state

Mamikon's Theorem: *The area of a tangent sweep is equal to the area of its tangent cluster, regardless of the shape of the original curve.*

You can see this in a real-world illustration when a bicycle's front wheel traces out one curve while the rear wheel (at constant distance from the front wheel) traces out another curve, as below. To find the area of the region between the two curves with calculus, you would need equations for both curves, but we don't need any here. The area of the tangent sweep is equal to the area of a circular sector depending only on the length of the bicycle and the change in angle from its initial position to its final position, as shown in the tangent cluster to the right. The shape of the bike's path does not matter.



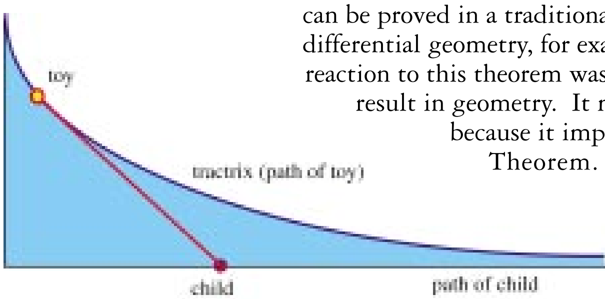
The next diagram illustrates the same idea in a more general setting. The only difference is that the tangent segments to the lower curve need not have constant length. We still have the tangent sweep (left) and the tangent cluster.



Mamikon's theorem, which seems intuitively obvious by now, is that the area of the tangent cluster is equal to the area of the tangent sweep. (To convince yourself, consider corresponding equal tiny triangles translated from the tangent sweep to the tangent cluster.)

In the most general form of Mamikon's theorem the given curve need not lie in a plane. It can be any smooth curve in space, and the tangent segments can vary in length. The tangent sweep will lie on a developable surface, one that can be rolled out flat onto a plane without distortion. The shape of the tangent sweep depends on how the lengths and directions of the tangent segments change along the curve; the tangent cluster lies on a conical surface whose vertex is the common point. Mamikon's general theorem equates the area of the tangent sweep with that of its tangent cluster.

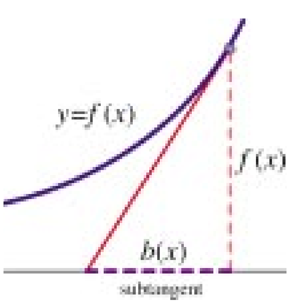
General Form of Mamikon's Theorem: *The area of a tangent sweep to a space curve is equal to the area of its tangent cluster.*



This theorem, suggested by geometric intuition, can be proved in a traditional manner—by using differential geometry, for example. My first reaction to this theorem was, "OK, that's a cool result in geometry. It must have some depth because it implies the Pythagorean Theorem. Can you use it to do anything else that's interesting?"

It turns out that you can apply this theorem in all sorts of interesting ways. As already mentioned, curves swept out by tangent segments of constant length include oval rings and the bicycle-tire tracks. Another such example is the tractrix, the trajectory of a toy on a taut string being pulled by a child walking in a straight line, as shown above. To find the area of the region between the tractrix and the x-axis using calculus, you have to find the equation of the tractrix. This in itself is rather challenging—it requires solving a differential equation. Once you have the equation of the tractrix, you have to integrate it to get the area. This also can be done, but the calculation is somewhat demanding; the final answer is simply $\pi L^2/4$, where L is the length of the string. But we can see that the tractrix is a particular case of the "bicyclix," so its swept area is given by a circular sector, and its full area is a quarter of a circular disk.

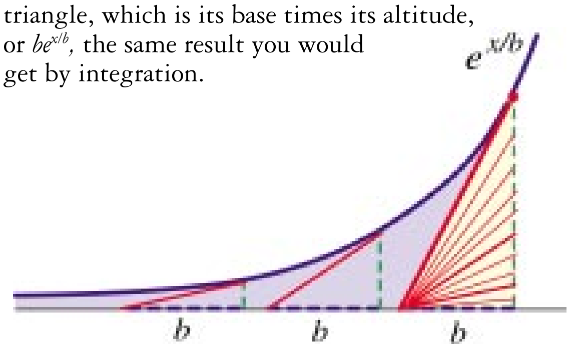
All the examples with tangents of constant length reveal the striking property that the area of the tangent cluster can be expressed in terms of the area of a circular sector without using any of the formal machinery of traditional calculus.



But the most striking applications are to examples in which the tangent segments are of variable length. These examples reveal the true power of Mamikon's method. This brings us to Problem 1: exponential curves. Exponential functions are ubiquitous in the applications of mathematics. They occur in problems concerning population growth, radioactive decay, heat flow, and other physical situations where the rate of growth of a quantity is proportional to the amount present. Geometrically, this means that the slope of the tangent line at each point of an exponential curve is proportional to the height of the curve at that point. An exponential curve can also be described by its subtangent, which is the projection of the tangent on the x-axis. The diagram at the bottom of the left-hand column shows a general curve with a tangent line and the subtangent. The slope of the tangent is the height divided by the length of the subtangent. So, the slope is proportional to the height if and only if the subtangent is constant.

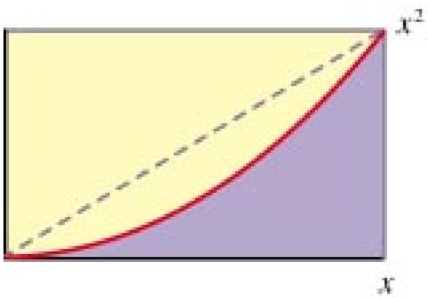
The next diagram, at the bottom of this column, shows the graph of an exponential curve $y = e^{x/b}$, where b is a positive constant. The only property of this curve that plays a role in this discussion is that the subtangent at any point has a constant length b. This follows easily from differential calculus, but it can also be taken as the defining property of the exponential. In fact, exponential curves were first introduced in 1684 when Leibniz posed the problem of finding all curves with constant subtangents. The solutions are the exponential curves.

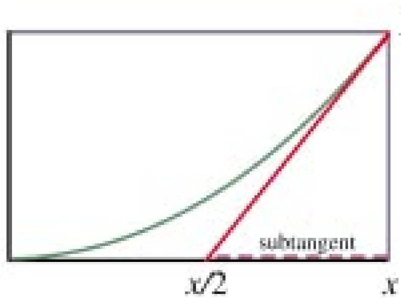
By exploiting the fact that exponential curves have constant subtangents, we can use Mamikon's theorem to find the area of the region under an exponential curve without using integral calculus. The diagram below shows the graph of the exponential curve $y = e^{x/b}$ together with its tangent sweep as the tangent segments, cut off by the x-axis, move to the left, from x all the way to minus infinity. The corresponding tangent cluster is obtained by translating each tangent segment to the right so that the endpoint on the x-axis is brought to a common point, in this case, the lower vertex of the right triangle of base b and altitude $e^{x/b}$. The resulting tangent cluster is the triangle of base b and altitude $e^{x/b}$. Therefore the area of the blue region is equal to the area of the yellow right triangle, so the area of the region between the exponential curve and the interval (from minus infinity to x) is equal to twice the area of this right triangle, which is its base times its altitude, or $be^{x/b}$, the same result you would get by integration.



This yields the astonishing result that the area of the region under an exponential curve can be determined in an elementary geometric way without the formal machinery of integral calculus.

We turn now to our second problem, perhaps the oldest calculus problem in history—finding the area of a parabolic segment, the purple region at left, below. The parabolic segment is inscribed in a rectangle of base x and altitude x^2. The area of the rectangle is x^3. From the figure we see that the area of the parabolic segment is less than half that of the rectangle in which it is inscribed. Archimedes made the stunning discovery that the area is exactly one-third that of the rectangle. Now we will use Mamikon's theorem to obtain the same result by a method that is not only simpler than the original treatment by Archimedes but also more powerful because it can be generalized to higher powers.



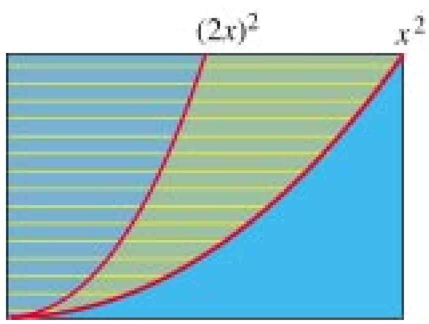


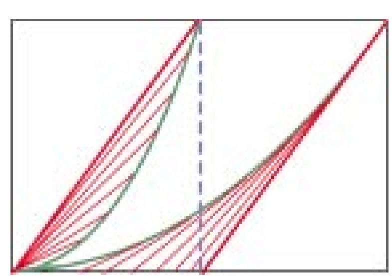
This parabola has the equation $y = x^2$, but we shall not need this formula in our analysis. We use only the fact that the tangent line above any point x cuts off a subtangent of length $x/2$, as indicated in the lower diagram. The slope of the tangent is x^2 divided by $x/2$, or $2x$.

To calculate the area of the parabolic segment we look at the next figure in which another parabola $y = (2x)^2$ has been drawn, exactly half as wide as the given parabola. It is formed by bisecting each horizontal segment between the original parabola and the y axis. The two parabolas divide the rectangle into three regions, and our strategy is to show that all three regions have equal area. If we do this, then each has an area one-third that of the circumscribing rectangle, as required.

The two shaded regions formed by the bisecting parabola obviously have equal areas, so to complete the proof we need only show that the region above the bisecting parabola has the same area as the parabolic segment below the original parabola. To do this, let's look at the next diagram, below. The right triangles here have equal areas (they have the same altitude and equal bases). Therefore the problem reduces to showing that the two shaded regions in this diagram have equal areas. Here's where we use Mamikon's theorem.

The shaded portion under the parabola $y = x^2$ is the tangent sweep obtained by drawing all the





With Mamikon's help, the Montessori schoolchildren built this 60-foot-long suspension bridge. Using themselves as weight, they are illustrating how heavy loading changes the shape of the cable from a catenary (the curve normally formed by hanging a chain from both ends) to a parabola. Such a "breakdown" occurred during the 50th-anniversary celebration of the Golden Gate Bridge in 1987.

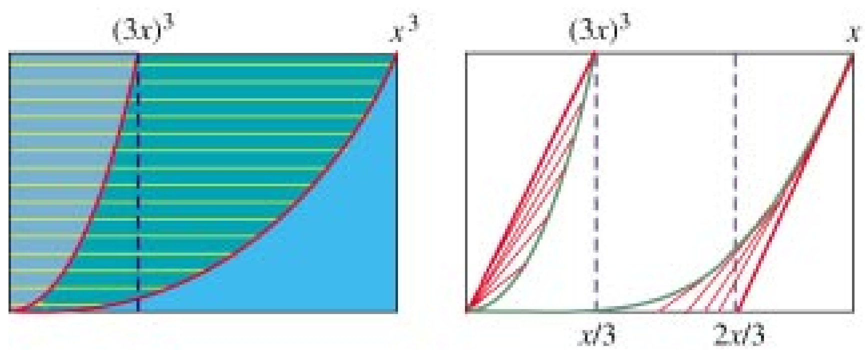
tangent lines to the parabola and cutting them off at the x-axis. And the other shaded portion is its tangent cluster, with each tangent segment translated so its point of intersection with the x axis is brought to a common point, the origin.

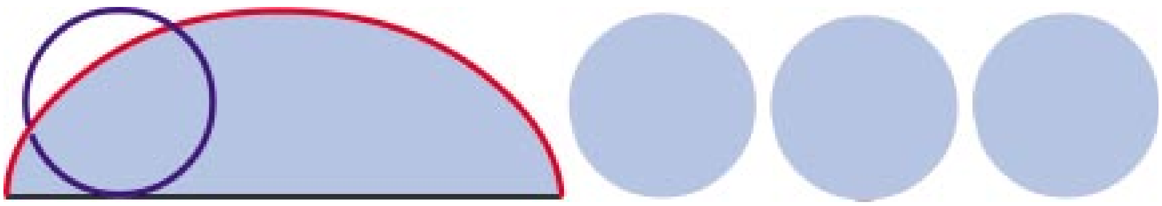
At a typical point (t, t^2) on the lower parabola, the tangent intersects the x-axis at $t/2$. Therefore, if the tangent segment from $(t/2, 0)$ to (t, t^2) is translated left by the amount $t/2$, the translated segment joins the origin and the point $(t/2, t^2)$ on the curve $y = (2x)^2$. So the tangent cluster of the tangent sweep is the shaded region above the curve $y = (2x)^2$, and by Mamikon's theorem the two shaded regions have equal areas, as required. So we have shown that the area of the parabolic segment is exactly one-third that of the circumscribing rectangle, the same result obtained by Archimedes.

The argument used to derive the area of a parabolic segment also extends to generalized parabolic segments, in which x^2 is replaced by higher powers. The graphs of $y = x^3$ and $y = (3x)^3$ at left divide the rectangle of area x^4 into three regions. The curve $y = (3x)^3$ *trisects* each horizontal segment in the figure, hence the area of the region above this curve is half that of the region between the two curves. In this case we will show that the area of the region above the trisecting curve is equal to that below the original curve, which means that each region has an area one-fourth that of the circumscribing rectangle.

To do this we use the fact that the subtangent is now one-third the length of the base, as shown below. One shaded region is the tangent sweep of the original curve, and the other is the corresponding tangent cluster, so they have equal areas. The right triangles are congruent, so they have equal areas. Therefore the region above the trisecting curve has the same area as the region below the curve $y = x^3$, and each is one-fourth that of the rectangle, or $x^4/4$. The argument also extends to all higher powers, a property not shared by Archimedes' treatment of the parabolic segment. For the curve $y = x^n$ we use the fact that the subtangent at x has length x/n.

We turn next to our third standard calculus

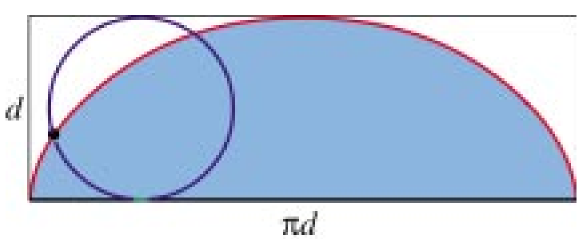




problem—the cycloid, the curve traced out by a point on the perimeter of a circular disk that rolls without slipping along a horizontal line. We want to show that the area of the region between one arch of the cycloid and the horizontal line is three times the area of the rolling disk (above), without deriving an equation for the cycloid or using integral calculus.

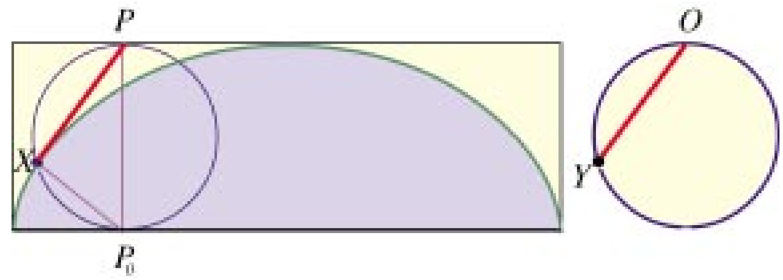
Below is a cycloidal arch inscribed inside a rectangle whose altitude is the diameter *d* of the disk and whose base is the disk's circumference, πd. The area of the circumscribing rectangle is πd^2, which is four times the area of the disk. So it suffices to show that the unshaded region above the arch and inside the rectangle has an area equal to that of the disk.

To do this, we show that the unshaded region is



the tangent sweep of the cycloid, and that the corresponding tangent cluster is a circular disk of diameter *d*. By Mamikon's theorem, this disk has the same area as the tangent sweep. Because the area of the disk is one-fourth the area of the rectangle, the area of the region below the arch must be three-fourths that of the rectangle, or three times that of the rolling disk.

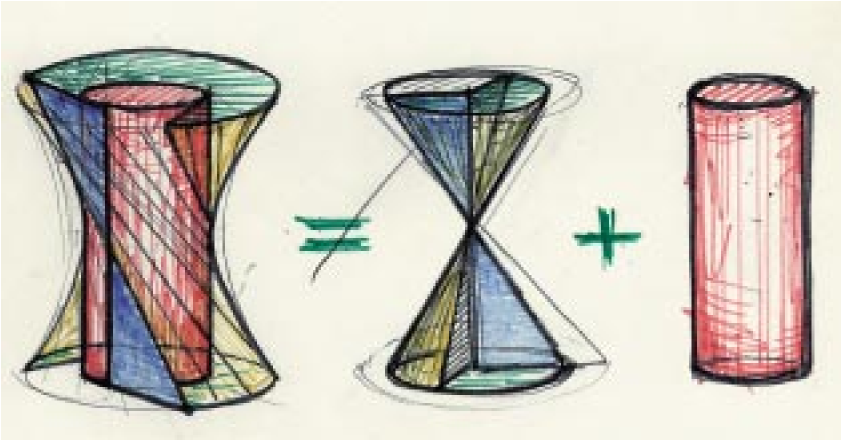
It remains to show that the tangent cluster of the unshaded region is a circular disk, as asserted. As the disk rolls along the base it is always tangent to the upper and lower boundaries of the circumscribing rectangle. If we denote the upper point of tangency by *P* and the lower point of



tangency by P_0, as in the diagram above, the diameter PP_0 divides the rolling circle into two semicircles, and any triangle inscribed in these semicircles must be a right triangle. The disk undergoes instantaneous rotation about P_0, so the tangent to the cycloid at any point *X* is perpendicular to the instantaneous radius of rotation and therefore must be a vertex of a right triangle inscribed in the semicircle with diameter PP_0. Consequently, the chord *XP* of the rolling disk is always tangent to the cycloid.

Extend the upper boundary of the circumscribing rectangle beyond the arch and choose a fixed point *O* on this extended boundary. Translate each chord parallel to itself so that point *P* is moved horizontally to the fixed point *O*. Then the other extremity *X* moves to a point *Y* such that segment *OY* is equal in length and parallel to *PX*. Consequently, *Y* traces out the boundary of a circular disk of the same diameter, with *OY* being a chord equal in length and parallel to chord *PX*. Therefore the tangent cluster is a circular disk of the same diameter as the rolling disk, and Mamikon's theorem tells us that its area is equal to that of the disk.

These examples display a wide canvas of geometric ideas that can be treated with Mamikon's methods, but seeing them static on a printed page leaves something to be desired. Animation, clearly, is a better way to show how the method works. So we plan to use these examples in the first of a series of contemplated videotapes under



One of Mamikon's 1959 hand sketches illustrates how the volume of a hyperboloid can be seen as disected into an inscribed cylinder and a "tangential" cone (the tangents to the cylinder). Right: Apostol in 1977.

the umbrella of *Project MATHEMATICS!* Like all videotapes produced by *Project MATHEMATICS!*, the emphasis will be on dynamic visual images presented with the use of motion, color, and special effects that employ the full power of television to convey important geometric ideas with a minimal use of formulas. The animated sequences will illustrate how tangent sweeps are generated by moving tangent segments, and how the tangent segments can be translated to form tangent clusters. They will also show how many classical curves are naturally derived from their intrinsic geometric and mechanical properties.

Mamikon's methods are also applicable to many plane curves not mentioned above. In subsequent videotapes we plan to find full and partial areas of the ellipse, hyperbola, catenary, logarithm, cardioid, epicycloids, hypocycloids, involutes, evolutes, Archimedean spiral, Bernoulli lemniscate, and sines and cosines. And we can find the volumes of three-dimensional figures such as the ellipsoid, the paraboloid, three types of hyperboloids, the catenoid, the pseudosphere, the torus, and other solids of revolution.

I'll conclude with a small philosophical remark: Newton and Leibniz are generally regarded as the discoverers of integral calculus. Their great contribution was to unify work done by many other pioneers and to relate the process of integration with the process of differentiation. Mamikon's method has some of the same ingredients, because it relates moving tangent segments with the areas of the regions swept out by those tangent segments. So the relation between differentiation and integration is also embedded in Mamikon's method. □

Samples of the computer animation of the problems shown in this article can be viewed at http://www.its.caltech.edu/~mamikon/calculus.html

Professor of Mathematics, Emeritus, Tom Apostol joined the Caltech faculty in 1950. On October 4, 2000, a special mathematics colloquium was held in honor of his 50 years at Caltech. On that occasion he delivered the talk that's adapted here. (Mamikon Mnatsakanian was also on hand to show his computer animations.)

Apostol earned his BS in chemical engineering (1944) and MS in mathematics (1946) from the University of Washington. His PhD, with a thesis in analytic number theory, is from UC Berkeley (1948). Before beginning his 50 years at Caltech, he spent a year each at Berkeley and MIT.

Apostol should know everything there is to know about teaching calculus, even though he admits he was surprised by this new approach. For nearly four decades Caltech undergraduates (as well as a couple of generations of mathematics students all over the country) have learned calculus from his two-volume text, often referred to as "Tommy 1" and "Tommy 2." These and his other textbooks in mathematical analysis and analytic number theory have been translated into Greek, Italian, Spanish, Portuguese, and Farsi.

Although known nationally and internationally for his written textbooks, Apostol turned to the visual media in the 1980s as a member of the Caltech team that produced The Mechanical Universe . . . and Beyond, *a 52-episode telecourse in college physics. And he never looked back. He's currently creator, director, and producer of* Project MATHEMATICS!, *a series of award-winning, computer-animated videotapes that are used nationwide and abroad as support material in high-school and community-college classrooms.*

PICTURE CREDITS:
22 — Bob Paz;
23–31 — Mamikon Mnatsakanian; 25, 27, 28–29 — Steven Hempel