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Area & Arc Length of Trochogonal Arches

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For the average lay person the word roulette means a gambling game, or perhaps a small toothed wheel that makes equally spaced perforations like those on sheets of postage stamps. In geometry a roulette is the locus of a point attached to the plane of a curve that rolls along a fixed curve. A surprising number of classical curves can be generated as roulettes: the cycloid, tractrix, catenary, parabola, ellipse, to name just a few. If the rolling curve is a circle, the roulette is called a trochoid (from τροχός the Greek word for wheel). This article treats generalized trochoids, called trochogons—curves traced by a point attached to a polygonal disk rolling along a line or around a fixed polygonal disk.

The special case of a regular polygon of n sides rolling around another regular polygon of m sides is of particular interest. If the edges of the two regular polygons have the same length, a point z attached rigidly to the n-gon traces out an arch consisting of n circular arcs before repeating the pattern periodically. In [3] we gave a simple method (without using calculus) for determining the area A of the region between the trochogonal arch and the fixed polygon. We call this the area of the trochogonal arch. It is given by

(1) $$A = P + \left(1 \pm \frac{n}{m}\right)(C + C_z),$$

where P denotes the area of the rolling n-gon, C is the area of the disk that circumscribes the n-gon, and C_z is the area of a disk whose radius is the distance from the center of the rolling n-gon to the tracing point z. The plus sign is used for epitrochogons (where the rolling n-gon is outside the fixed m-gon), and the minus sign for hypotrochogons (where the rolling n-gon is inside the m-gon).

When the tracing point z is at a vertex of the rolling polygon, the terms epicyclogon and hypocyclogon are used instead of epitrochogon and hypotrochogon. The main result in this article is the following formula for their arc lengths:

(2) $$L = 4D\left(1 \pm \frac{n}{m}\right)\left(\frac{\pi}{2n}\cot\frac{\pi}{2n}\right),$$

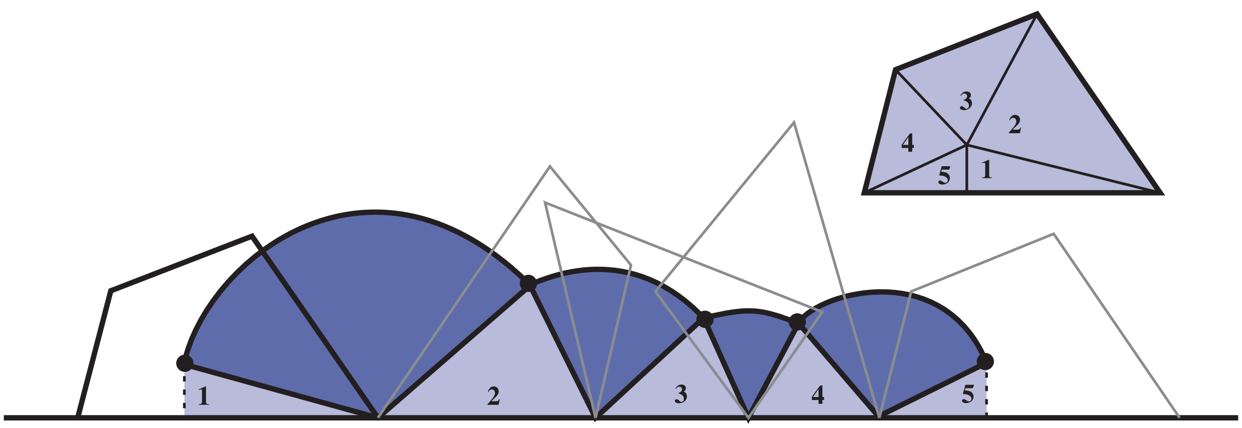

Figure 1. A trochogon traced out by a point in a quadrilateral disk rolling along a line.

where D is the diameter of the circle that circumscribes the rolling n-gon.

A General n-gon Rolling Along a Line

We consider first a convex n-gon (not necessarily regular) rolling along a fixed straight line. The rolling quadrilateral in Figure 1 displays all the essential features required for treating a general n-gon.

Here the tracing point z is inside the quadrilateral, and in one revolution it sweeps out four circular arcs. The trochogonal region below these arcs and above the fixed line consists of four circular sectors and a set of triangles that form a dissection of the quadrilateral as indicated in Figure 1.

In the general case of an n-gon making one revolution along a line, the tracing point z generates a trochogonal region consisting of n circular sectors together with a set of triangles that provide a dissection of the n-gon. The area A of this region is equal to the area P of the rolling n-gon, plus the sum of the areas of n circular sectors. The kth sector has area $(1/2)\phi_k r_k^2$, where $r_1,\ldots, r_n$, are the radii of the sectors and ϕ_k is the angle (in radians) subtended by the sector of radius r_k. Note that radius r_k is the distance from the tracing point z to the kth vertex of the rolling polygon, and angle ϕ_k is the exterior angle of the polygon at that vertex. Thus we have

$$A = P + \frac{1}{2}\sum_{k=1}^{n}\phi_k r_k^2. \tag{3}$$

The length of the circular arc subtended by the kth sector is $\phi_k r_k$. The sum of these lengths is the length L of one arch of the trochogon:

$$L = \sum_{k=1}^{n}\phi_k r_k. \tag{4}$$

The sums on the right of (3) and (4) cannot be simplified until more is known about the radii and the exterior angles. The next three sections treat regular rolling polygons.

Regular n-gon Rolling Along a Line

When the rolling n-gon is regular, the trochogon is called a cyclogon if the tracing point is at a vertex, a curtate cyclogon if the tracing point is inside the polygon, and a prolate cyclogon if it is outside. In this case each exterior angle ϕ_k is equal to $2\pi/n$ and the respective formulas (3) and (4) become

$$A = P + \frac{\pi}{n}\sum_{k=1}^{n} r_k^2 \tag{5}$$

and

$$L = \frac{2\pi}{n}\sum_{k=1}^{n} r_k. \tag{6}$$

The sum in (5) can be simplified with the help of a theorem on sums of squares derived in [2]. In complex number notation, it states that if $z_1, z_2, \ldots, z_n$ lie on a circle of radius r with center at the origin 0, and if the centroid of these points is also at 0, then for any point z in the same plane we have

$$\sum_{k=1}^{n}|z - z_k|^2 = n\left(r^2 + |z|^2\right). \tag{7}$$

If the points $z_1, z_2, \ldots, z_n$ are the vertices of a regular n-gon with center at 0, we can apply (7) with

$$r_k = |z - z_k|$$

and we find

$$\frac{\pi}{n}\sum_{k=1}^{n} r_k^2 = \pi r^2 + \pi|z|^2 = C + C_z$$

where C is the area of the disk that circumscribes the polygon, and C_z is the area of a disk whose radius is the distance from the center of the rolling disk to the tracing point z. Using this in (5) we find the simple and elegant result

$$A = P + C + C_z \tag{8}$$

which agrees with (1) when m goes to infinity.

There is no analog of (7) that can be used in (6) to obtain a simple arc length formula analogous to (8). This is not surprising because calculating arc length is usually more difficult than calculating area. For example, the area of the region enclosed by any ellipse can be easily determined (with or without calculus), but calculating the arc length of a general ellipse requires elliptic integrals. In the next two sections we find (without using calculus) the arc lengths of epi- and hypocyclogonal arches when the tracing point z is at a vertex of the rolling regular polygon.

The Arc Length of a Cyclogon

When the tracing point z is at a vertex of the regular n-gon, the trochogon is a cyclogon, and area formula (8) becomes

$$A = P + 2C \tag{9}$$

which was first given in [1]. In the limiting case when $n \to \infty$, area P becomes C, the cyclogon becomes a cycloid, and (9) gives us the classical result $A = 3C$. Another classical result states that the length L of a cycloidal arch is equal to $4D$, where D is the diameter of the rolling circle. In this section we generalize this result by showing that the arc length of a cyclogon generated by a rolling n-gon is given by

$$L = 4D\left(\frac{\pi}{2n}\cot\frac{\pi}{2n}\right), \tag{10}$$

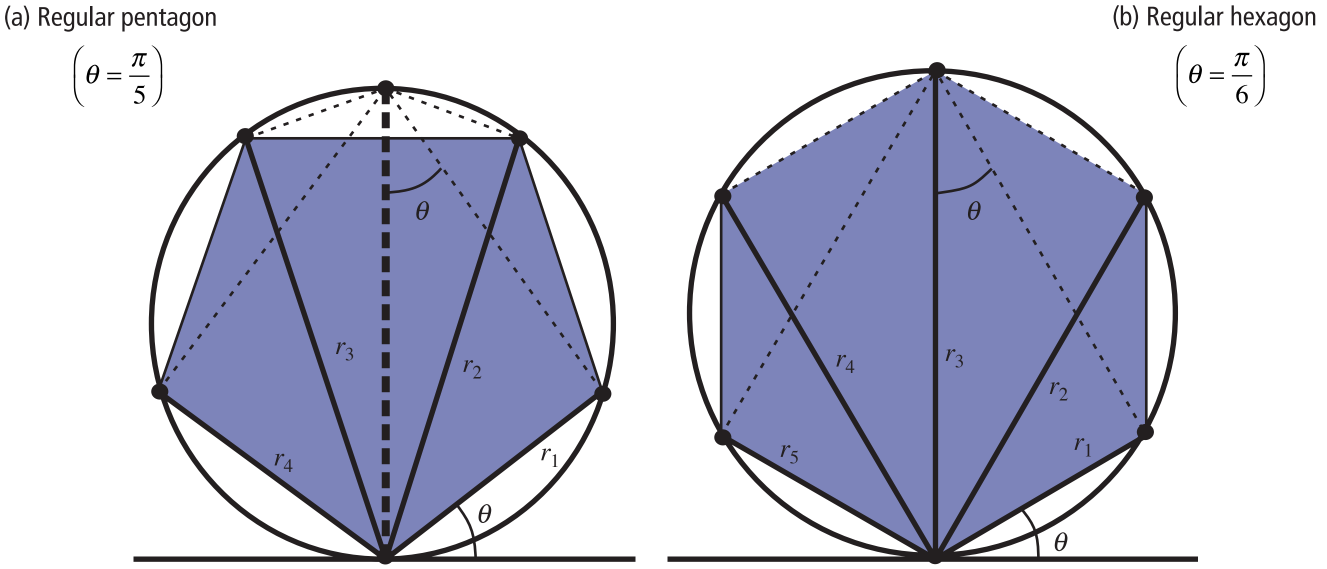

Figure 2. Segments drawn from one vertex of a regular polygon to the remaining vertices.

where D is the diameter of the circle that circumscribes the n-gon. This implies that $L \to 4D$ as $n \to \infty$ because

$$\lim_{n\to\infty}\left(\frac{\pi}{2n}\cot\frac{\pi}{2n}\right) = 1.$$

The quantity

$$\frac{\pi}{2n}\cot\frac{\pi}{2n}$$

is surprisingly close to 1, even for small values of n. For example, for $n = 3, 4, 5, 6$ its values are 0.91, 0.95, 0.97, 0.98. Note also that (10) is the limiting case of (2) when $m \to \infty$.

To obtain (10) we start with (6) and note that radius r_k is the distance from the tracing point to the kth vertex of the polygon. When the tracing point is one of the vertices, formula (6) becomes

$$L = \frac{2\pi}{n}\sum_{k=1}^{n-1} r_k, \tag{11}$$

where now $r_1, \ldots, r_{n-1}$ are the lengths of the segments from one vertex to each of the remaining $n - 1$ vertices. Figure 2 shows two examples, (a) a regular pentagon, and (b) a regular hexagon. In (a) there are four segments r_1, r_2, r_3, r_4 symmetrically located about a diameter of length D. In (b) there are five segments, one of which is a diameter, the other four being located symmetrically about this diameter.

In a general regular n-gon, each segment r_k together with a diameter D determines a right triangle inscribed in a semicircle with the diameter as hypotenuse. One of the acute angles of the right triangle is $k\theta = k\pi/n$, so $r_k = D \sin k\pi/n$, and (11) becomes

$$L = \frac{2\pi D}{n}\sum_{k=1}^{n-1}\sin\frac{k\pi}{n}. \tag{12}$$

Fortunately there is a trigonometric identity that is tailor-made to evaluate the sum of sines in (12). It states that for any real x not an integer multiple of π we have

$$\sum_{k=1}^{n}\sin(2kx) = \frac{\sin(n+1)x\,\sin nx}{\sin x}. \tag{13}$$

This identity is a disguised form of an elementary formula for the sum of a geometric progression:

$$\sum_{k=1}^{n} z^k = z\frac{z^n - 1}{z - 1}, \tag{14}$$

valid for any complex $z \neq 1$. To deduce (13), take $z = e^{2ix}$ in (14) and we find

$$\begin{aligned}\sum_{k=1}^{n} e^{2ikx} &= e^{2ix}\frac{e^{2inx} - 1}{e^{2ix} - 1}\\ &= e^{2ix}\frac{e^{inx}\left(e^{inx} - e^{-inx}\right)}{e^{ix}\left(e^{ix} - e^{-ix}\right)}\\ &= e^{i(n+1)x}\frac{\sin nx}{\sin x}.\end{aligned}$$

Now equate imaginary parts of this last equation to obtain (13). When $x = \pi/(2n)$, the term with $k = n$ in equation (13) vanishes and we get

$$\sum_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{\sin\left(\frac{\pi}{2n} + \frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right)}{\sin \frac{\pi}{2n}} = \cot \frac{\pi}{2n},$$

and we see that (12) becomes (10), which gives the arc length of every cyclogon.

Arc Length of Epicyclogons and Hypocyclogons

The curve traced out by a point on a circle rolling along a fixed circle is called an epicycloid if the rolling circle is outside the fixed circle, and a hypocycloid if it is inside. This section treats the more general case of a regular n-gon rolling along a fixed regular m-gon, where the edges of the two polygons have equal length. The corresponding curves traced out by a vertex of the n-gon are called epicyclogons and hypocyclogons. The analysis used in the previous section can be applied to this more general case. In equation (4) the angle ϕ_k through which the polygon rolls about the kth vertex is equal to $2\pi/n + 2\pi/m$ for the epicyclogon, and $2\pi/n - 2\pi/m$ for the hypocyclogon. For the length of one arch we have, instead of (6), the relation

(15) $$L = \left(\frac{2\pi}{n} \pm \frac{2\pi}{m}\right)\sum_{k=1}^{n} r_k,$$

where the plus sign is used for the epicyclogon and the minus sign for the hypocyclogon. Once again we have $r_k = D \sin k\pi/n$, where D is the diameter of the circle that circumscribes the rolling polygon, and instead of (10) we obtain the result announced in the introduction:

(2) $$L = 4D\left(1 \pm \frac{n}{m}\right)\left(\frac{\pi}{2n}\cot\frac{\pi}{2n}\right).$$

We can obtain the limiting case of a circle of radius r rolling around a fixed circle of radius R if we let both n and m tend to infinity in such a way that their ratio $n/m \to r/R$. Then the limiting value of (2) gives the lengths of the corresponding epicycloid and hypocycloid. If we note that the diameter D of the rolling circle is $2r$, we find the limiting value of the arc length is

$$L = 4D\left(1 \pm \frac{D}{2R}\right) = 8r\left(1 \pm \frac{r}{R}\right).$$

Examples

Some classic trochoids have names suggested by their shapes, for example, cardioid, nephroid, astroid and deltoid. We have given similar names to the trochogons having these trochoids as limits. For example, a cardiogon is generated by a regular n-gon rolling around a fixed copy of itself. Figure 3 shows an example with $n = 6$. As $n \to \infty$ the cardiogon becomes a cardioid. A thorough discussion of the history of many classic curves can be found on the web site `http://www-groups.dcs.st-andrews.ac.uk/history/Curves`, where one can also see superb animation related to these curves. Another rich source is the handbook by Yates [4], which also contains formulas for areas and arc lengths.

Table 1 gives arc lengths and areas of one arch of a few special trochogons as derived from (2) and (1), together with known

epi/hypocyclogon	Factor $1 \pm n/m$	Limiting value of arc length	Area	epi/hypocycloid	Area
cyclogon	$1 + n/\infty = 1$	$4D$	$P + 2C$	cycloid	$3C$
cardiogon	$1 + n/m = 2$	$8D$	$P + 4C$	cardioid	$5C$
nephrogon	$1 + n/m = 3/2$	$6D$	$P + 3C$	nephroid	$4C$
astrogon	$1 - n/m = 3/4$	$3D$	$P + 3C/2$	astroid	$5C/2$
deltogon	$1 - n/m = 2/3$	$8D/3$	$P + 4C/3$	deltoid	$7C/3$
diamogon	$1 - n/m = 1/2$	$2D$	$P + C$	diameter	$2C$

Table 1. Arc lengths and areas of one arch of some special trochogons, and classical results for the corresponding trochoids.

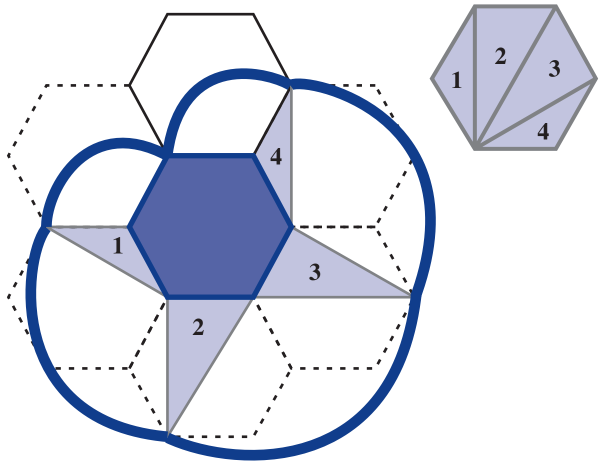

Figure 3. A cardiogon generated by a regular hexagon rolling around a fixed copy of itself.

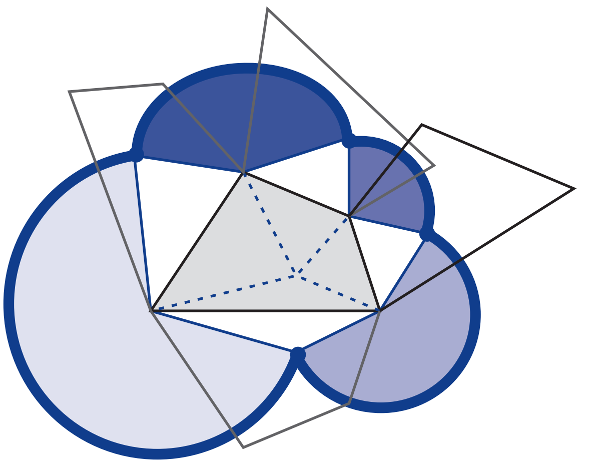

Figure 4. Autogon generated by the quadrilateral in Figure 1 rolling around a fixed copy of itself.

classical results for the corresponding trochoids. We recall that P is the area of the rolling regular n-gon, C is the area of the circumscribing circle, and D is its diameter.

The last entry may seem somewhat surprising. What we call a diamogon is a hypocyclogon with $n/m = 1/2$. The curve is traced by a point z at a vertex of an n-gon rolling inside a $2n$-gon. When the n-gon makes one circuit around the inside of the $2n$-gon, it traces out two curves, each consisting of $n - 1$ circular arcs situated symmetrically about a diameter of the $2n$-gon. Using Eq. (2) we find that the length of one arch of the diamogon is

$$2D\left(\frac{\pi}{2n}\cot\frac{\pi}{2n}\right).$$

When $n \to \infty$ this becomes $L = 2D$, the diameter of the fixed circle. This is consistent with the fact that when $n \to \infty$ one arch of the diamogon turns into a diameter of the fixed circle. The region between the arch and the fixed circle is a semi-circular disk of area $2C$, half the area of the fixed circle.

Area and Arc Length of Autogons

We return to an arbitrary n-gon (not necessarily regular) that rolls around a fixed copy of itself, so that in one revolution each edge is made to coincide with a congruent edge. A point z rigidly attached to the rolling n-gon traces out a curve we call an autogon. Figure 3 shows an autogon traced by the vertex of a regular hexagon. Figure 4 shows an autogon traced by the point z inside the nonregular quadrilateral that appears in Figure 1.

For a general autogon the turning angle at each vertex is twice that of the exterior angle ϕ_k that appears in rolling along a line. Therefore the general formulas for area and arc length given in (3) and (4) now become

$$A_a = P + \sum_{k=1}^{n} \phi_k r_k^2 \tag{16}$$

and

$$L_a = 2\sum_{k=1}^{n} \phi_k r_k, \tag{17}$$

where the subscripts on the symbols A_a and L_a indicate that we are considering autogons. Although there is no general formula for simplifying the sums in (16) and (17), we can compare the area and arc length of an autogon with the corresponding trochogonal area A and arc length L as given in (3) and (4). Comparing (16) with (3) we see that $A_a - P = 2(A - P)$, so $A_a + P = 2A$. The geometric meaning of this relation is illustrated in Figure 4. The autogon consists of one closed arch surrounding the fixed polygon, so the entire area enclosed inside the autogon, $A_a + P$, is twice the area of the corresponding trochogon. Comparing (17) with (4) we also see that $L_a = 2L$, so the arc length of any autogon is twice that of the corresponding trochogon. So we have discovered the following comparison theorems.

Theorem 1. The area of the region inside an autogon is twice that of the corresponding trochogonal arch obtained by rolling the polygon along a line.

Theorem 2. The arc length of an autogon is twice that of the corresponding trochogonal arch obtained by rolling the polygon along a line.

Note that both theorems are valid for any location of the tracing point z. The theorems can be visualized geometrically by comparing the autogon in Figure 4 with the trochogon in Figure 1. Both are traced by the same point z inside the same

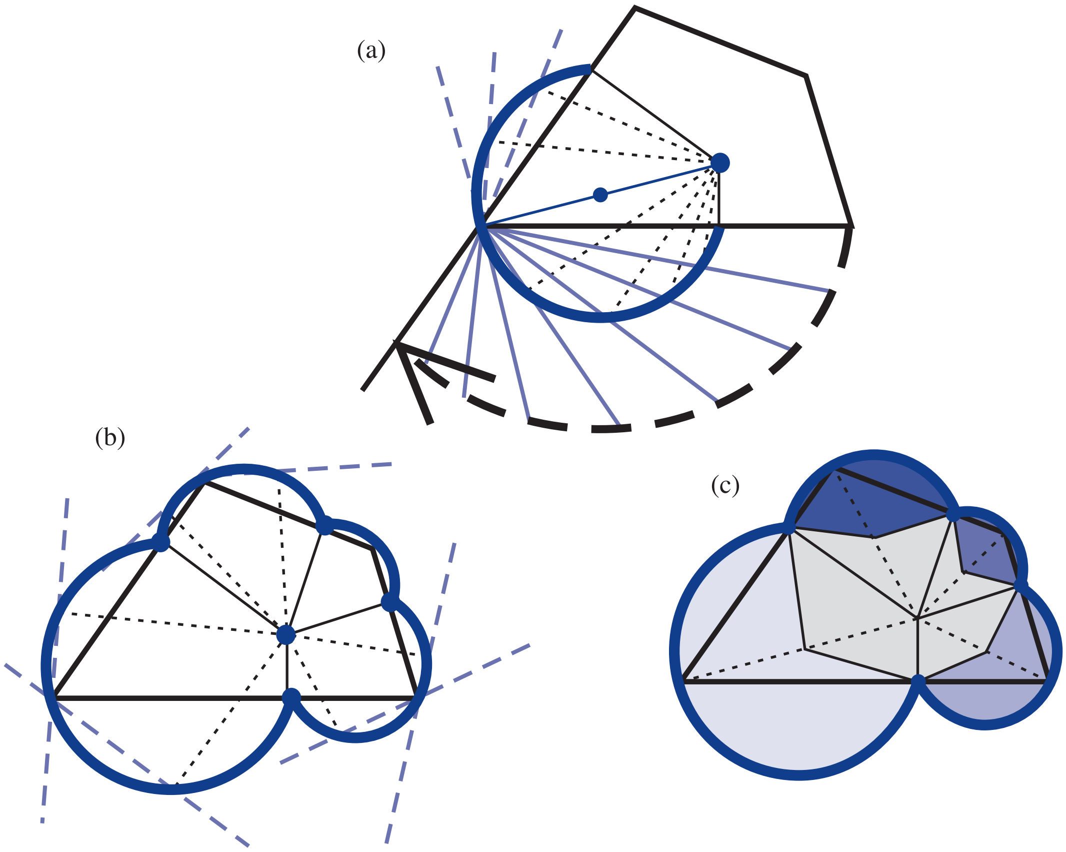

Figure 5. The pedal curve of a quadrilateral with respect to a pedal point z inside.

rolling quadrilateral. Each circular sector in Figure 4 subtends twice the angle of the corresponding sector in Figure 1, but has the same radius, hence twice the area and twice the arc length. The polygonal region surrounded by the sectors in Figure 4 contains the quadrilateral plus the four triangles that provide a dissection of the quadrilateral, so this polygonal region has area twice that of the quadrilateral.

The theorems also apply to any smooth curve rolling around a fixed copy of itself if that curve is the limit of n-gons as $n \to \infty$. In particular they apply to a circular disk rolling around a fixed equal disk. If the tracing point z is on the boundary of the disk, the roulette is called a cardioid. When the tracing point is not on the boundary the cardioid is called a limaçon of Pascal, which can be regarded as a curtate or prolate cardioid. Consequently we have the following corollaries:

Corollary 1. The area of the region inside a limaçon of Pascal (or a cardioid) is twice that of the corresponding curtate or prolate cycloidal arch (or cycloid) obtained by rolling the same disk along a line.

Corollary 2. The arc length of a limaçon of Pascal (or a cardioid) is twice that of the corresponding curtate or prolate cycloidal arch (or cycloid) obtained by rolling the same disk along a line.

The results for the limaçon of Pascal were obtained by the 19th century geometer Jakob Steiner by a different method that uses pedal curves. The next section relates autogons and pedal curves.

Pedal Curves and Steiner's Theorems

Some roulettes are pedal curves, which are defined as follows. Given a fixed smooth plane curve Γ and a fixed point z not on Γ called a *pedal point*, let p denote the foot of the perpendicular from z to a typical tangent line to Γ. The locus of all such points p constructed for all tangent lines to Γ is called the pedal curve of Γ with respect to z. Although it is not obvious, it can be shown that the pedal curve of a circle is a cardioid if the pedal point is on the circle, or a limaçon of Pascal if the pedal point is inside or outside the circle.

Steiner discovered a remarkable property relating the area of any roulette with that of its pedal curve, and another relating their arc lengths. Steiner's first theorem states that when a smooth closed curve Γ rolls along a straight line, the area of the region between one arch of the roulette traced by a point z attached to Γ and the straight line is twice the area of the region enclosed by the pedal curve of Γ with z as pedal point. His second theorem states that the arc length of one arch of the roulette is equal to the arc length of the pedal curve. We will deduce Steiner's theorems from Theorems 1 and 2. But first we obtain comparison theorems relating areas and arc lengths of autogons with those of their pedal curves.

Because a polygon has only a finite number of tangent lines (one for each edge) the usual definition of pedal curve would produce only a finite set of points as the pedal curve. So, we need to extend the concept of pedal curve so that it applies to polygons. The problem is to introduce a suitable replacement for a tangent line at a vertex of a polygon. We do this as follows.

Each vertex v of a polygon is the intersection of two consecutive edges. From a given pedal point z inside or outside the polygon, perpendiculars to these edges intersect the polygon at points p and q, say. Figure 5(a) shows an example with z inside. Imagine rotating a line through one edge about this vertex through the exterior angle at that vertex until it reaches the line of an adjacent edge. Each intermediate position of the rotating line can play the role of a tangent line, and a perpendicular can be drawn from the pedal point z to each such line. The locus of the feet of the perpendiculars will lie on a circle (passing through the common vertex) whose diameter d is the distance from z to that vertex. The circular arc joining p and q is, by definition, the portion of the pedal curve contributed by that vertex. So the pedal curve of any n-gon consists of n circular arcs, one for each vertex. The diameter of each arc is the distance from the pedal point to the corresponding vertex. The pedal curve of the quadrilateral in Figure 5(b) consists of four circular arcs. For a convex polygon, no point of the pedal curve lies inside the polygon.

Now take an arbitrary n-gon and roll it around a fixed mirror image so that in one revolution each edge is made to coincide with a congruent edge. A point z rigidly attached to the rolling n-gon traces out an autogon. The pedal curve to this autogon with pedal point z will be a scaled copy of the autogon with a scaling factor of 1/2. This fact is known for smooth curves but it holds for arbitrary n-gons as well. The reason is that each point on the pedal curve is the foot of a perpendicular from the pedal point z and therefore must be at the midpoint of the extended perpendicular that joins z to its mirror image. Consequently, we have the following comparison theorems for autogons and their pedal curves.

Theorem 3. The area enclosed by one arch of an autogon traced by z is four times that enclosed by the pedal curve having pedal point z.

Theorem 4. The arc length of an autogon traced by z is twice that of the pedal curve having pedal point z.

These can be visualized geometrically by comparing the shaded regions inside the pedal curve in Figure 5(c) with the autogon in Figure 4. The same point z traces the autogon in Figure 4 and is the pedal point in Figure 5. Each circular sector swept out by z in Figure 4 subtends the same angle as in Figure 5(b) but has twice the radius. So the area of the sector in Figure 4 is four times that in Figure 5(c), and its arc length is twice that in Figure 5(c). And the polygonal region surrounded by the circular sectors in Figure 4 has linear dimensions twice those in Figure 5(c), so its area is four times as large.

Now we use Theorems 3 and 4 together with Theorems 1 and 2 to deduce Steiner's theorems for smooth curves. By Theorem 1, the area of a trochogonal arch traced by a point z attached to an n-gon rolling along a straight line is half that of the corresponding autogon, hence, by Theorem 3, it is twice that of the pedal curve of the n-gon with pedal point z. This implies Steiner's first theorem for any smooth curve that is the limit of n-gons as $n \to \infty$. Steiner's second theorem follows similarly from Theorems 2 and 4.

The comparison Theorems 1 and 2 are simpler and more fundamental than Steiner's theorems. They are simpler because they imply Steiner's results for the limaçon of Pascal without the need of introducing pedal curves. And they are more fundamental because they apply to general autogons and can be used to deduce Steiner's theorems as a limiting case. ■

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