Given n fixed points in a plane, a point moving in the plane of these points in such a way that the sum of the squares of the distances from the points is constant traces out a circle whose center is at the centroid of the fixed points.

Sums of Squares of Distances

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A point that moves in a plane so that the sum of its distances from two fixed points in the plane is constant traces out an ellipse with the two fixed points as foci. What is the locus if the point moves so that the sum of the squares of the two distances is constant? An elementary calculation in coordinate geometry shows that the locus is a circle with center midway between the two fixed points.

When we ask the same question for three or more fixed points in a plane we obtain the following surprising result.

Theorem 1. *Given n fixed points in a plane, a point moving in the plane of these points in such a way that the sum of the squares of the distances from the points is constant traces out a circle whose center is at the centroid of the fixed points*.

Proof. Start with n arbitrary points in a plane, and let O denote their centroid. Using O as the origin of a complex plane, denote the given points by the n complex numbers $z_1, z_2, \ldots, z_n$, so that

$$\sum_{k=1}^{n} z_k = 0. \tag{1}$$

Now let z denote an arbitrary point in the plane, and consider the sum of the squares of the distances from z to each of the given points:

$$\sum_{k=1}^{n} |z - z_k|^2$$

The kth term of the sum is

$$(z - z_k)(\bar{z} - \bar{z}_k) = |z|^2 + |z_k|^2 - z\bar{z}_k - \bar{z}z_k.$$

Summing on k and using (1) we find

$$\begin{aligned} \sum_{k=1}^{n} |z - z_k|^2 &= n|z|^2 + \sum_{k=1}^{n} |z_k|^2 \\ &= n|z|^2 + nD_{n,}^2 \end{aligned} \tag{2}$$

where

$$D_n^2 = \frac{1}{n}\sum_{k=1}^{n} |z_k|^2$$

is the average of the squares of the distances of the points z_1, $z_2, \ldots, z_n$ from their centroid. Consequently,

$$\sum_{k=1}^{n} |z - z_k|^2$$

is constant if and only if $n|z|^2 + nD_n^2$ is constant. The set of all such z is a circle centered at the centroid O, if we allow the empty set and a single point to be considered as degenerate cases of a circle. ■

Professor Douglas Hofstadter has informed us that the special case of Theorem 1 in which there are three given points forming the vertices of a triangle can be found in Roger A. Johnson's 1929 *An Elementary Treatise on the Geometry of the Triangle and the Circle*, Corollary to Theorem 275.

The key to Theorem 1 is formula (2), which holds for any set of n + 1 points $z_1, z_2, \ldots, z_n$, and z for which the first n points satisfy (1). If $z_1, z_2, \ldots, z_n$ lie on a circle of radius R with center at their centroid O, formula (2) gives us

$$\sum_{k=1}^{n} |z - z_k|^2 = n\left(|z|^2 + R^2\right). \tag{3}$$

This holds, in particular if $z_1, z_2, \ldots, z_n$ are the vertices of a regular n-gon, or more generally if they are the vertices of a centrally symmetric polygon. If the point z also lies on the circle of radius R the sum in (3) reduces to

$$\sum_{k=1}^{n} |z - z_k|^2 = 2nR^2, \tag{4}$$

a generalization of the Pythagorean Theorem (which is the special case n = 2).

Another interesting special case occurs when z is one of the vertices. In this case, one term in the sum in (4) vanishes, and we obtain:

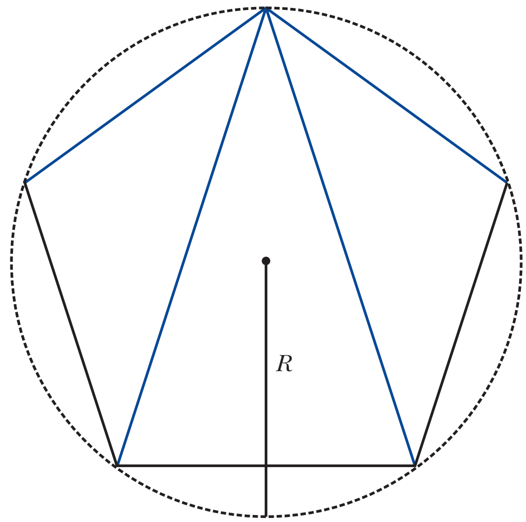
Theorem 2. *The sum of the squares of the $n - 1$ segments drawn from one vertex of a regular n-gon to the remaining vertices is equal to $2nR^2$, where R is the radius of the circumscribing circle.*

Theorem 2 was used in our article Cycloidal Areas Without Calculus (*Math Horizons*, Sept. 1999) to calculate cycloidal areas without calculus. The proof of Theorem 2 given in our earlier article treats separately polygons with an even and an odd number of sides. The authors were led to the more general formula (3) while seeking a proof that works for both odd and even n.

Incidentally, Theorems 1 and 2 and all related results hold in higher-dimensional spaces (with spheres instead of circles). A general proof can be given by using dot products for the position vectors z and z_k.

We conclude this note by using (4) to relate the length h of one edge of a regular simplex in k-space with the radius R of the circumscribing sphere. A regular k-dimensional simplex has $k + 1$ vertices, so if z is one of these vertices, one term in (4) vanishes and when $n = k + 1$ we get $kh^2 = 2(k + 1)R^2$, which gives us

$$h = \sqrt{2\left(1+\tfrac{1}{k}\right)}\, R.$$



The sum of the squares of the distances from one vertex to the other four vertices of a regular pentagon is equal to $10R^2$.

For a regular tetrahedron in 3-space ($k = 3$) we find $h = \sqrt{8/3}\,R$. This result, together with the law of cosines, enables us to find the angle between any two radii drawn from the center of the tetrahedron to the endpoints of one edge. The result is π–arccos1/3 (the corresponding result in k-space is π-arccos$1/k$). The distance h and the angle between two radii can also be calculated directly using elementary trigonometry but this requires much more effort. (Try it!) ■