Sums of Squares of Distances in m-Space

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1. INTRODUCTION. Given a finite set of fixed points in 3-space, what is the locus of a point moving in such a way that the sum of the squares of its distances from the fixed points is constant? The answer is both elegant and surprising: the locus is a sphere whose center is at the centroid of the fixed points (if we allow the empty set and a single point as degenerate cases of a sphere). The answer to this question and its extension to any finite-dimensional space was given in [**2**] and was based on equation (3′), which relates sums of squares of distances between points in m-space. This formula was used in [**1**], [**3**], and [**4**] to find, without calculus, the areas of cycloidal and trochoidal regions. It was also applied in [**2**] to problems regarding a regular simplex in k-space. This paper gives generalized formulas that lead to remarkable relations for sums of squares of distances in any finite-dimensional space. It also points out some applications to geometry.

2. BASIC THEOREMS. We begin with n arbitrary points $\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_n$, regarded as position vectors in Euclidean m-space. Let $w_1, w_2, \dots, w_n$ be n positive numbers regarded as *weights* attached to these points, and let $\boldsymbol{c}$ denote the weighted average position vector, defined by

$$\sum_{k=1}^{n} w_k \boldsymbol{r}_k = W\boldsymbol{c}, \tag{1}$$

where

$$W = \sum_{k=1}^{n} w_k \tag{2}$$

is the sum of the weights. Then we have:

Theorem 1. *If $\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_n$ are n points in m-space with weighted average $\boldsymbol{c}$ and if $\boldsymbol{z}$ is an arbitrary point, then the following holds:*

$$\sum_{k=1}^{n} w_k |\boldsymbol{z} - \boldsymbol{r}_k|^2 = \sum_{k=1}^{n} w_k |\boldsymbol{c} - \boldsymbol{r}_k|^2 + W|\boldsymbol{c} - \boldsymbol{z}|^2. \tag{3}$$

Proof. We give a simple proof using dot products. The kth term of the sum on the left is

$$w_k |\boldsymbol{z} - \boldsymbol{r}_k|^2 = w_k (\boldsymbol{z} - \boldsymbol{r}_k) \cdot (\boldsymbol{z} - \boldsymbol{r}_k) = w_k (|\boldsymbol{z}|^2 + |\boldsymbol{r}_k|^2 - 2\boldsymbol{z} \cdot \boldsymbol{r}_k).$$

Similarly, the kth term of the sum on the right is

$$w_k |\boldsymbol{c} - \boldsymbol{r}_k|^2 = w_k (|\boldsymbol{c}|^2 + |\boldsymbol{r}_k|^2 - 2\boldsymbol{c} \cdot \boldsymbol{r}_k),$$

and their difference is

$$w_k|\boldsymbol{z}-\boldsymbol{r}_k|^2 - w_k|\boldsymbol{c}-\boldsymbol{r}_k|^2 = w_k(|\boldsymbol{z}|^2 - |\boldsymbol{c}|^2) + 2w_k(\boldsymbol{c}-\boldsymbol{z})\cdot\boldsymbol{r}_k.$$

Summing on k and using (1) and (2), we find that

$$\sum_{k=1}^{n} w_k|\boldsymbol{z}-\boldsymbol{r}_k|^2 - \sum_{k=1}^{n} w_k|\boldsymbol{c}-\boldsymbol{r}_k|^2 = W(|\boldsymbol{z}|^2 - |\boldsymbol{c}|^2) + 2W(\boldsymbol{c}-\boldsymbol{z})\cdot\boldsymbol{c}$$

$$= W(|\boldsymbol{z}|^2 + |\boldsymbol{c}|^2 - 2\boldsymbol{z}\cdot\boldsymbol{c}) = W|\boldsymbol{c}-\boldsymbol{z}|^2,$$

which gives (3). ■

When $m = 2$, Theorem 1 is related to a result in classical mechanics called Steiner's parallel-axis theorem, which states that the moment of inertia of a rigid body about an arbitrary axis is equal to the moment of inertia about a parallel axis through the center of mass of the body, plus the mass times the square of the distance between the two axes. Theorem 1 can be regarded as both a discrete analog of the parallel-axis theorem and an extension of it to higher dimensional space.

Sums of the type considered here also occur in physics problems related to kinetic energy of a system of particles (where the vectors $\boldsymbol{r}_k$ represent velocities), and in angular momentum problems (where the $\boldsymbol{r}_k$ represent position vectors and the w_k are proportional to the mass times the angular speed of the particle). They also occur in probability theory. If weight w_k represents the probability that a random variable takes the value $\boldsymbol{r}_k$, then $\boldsymbol{c}$ is the expectation of the random variable and

$$\sum_{k=1}^{n} w_k|\boldsymbol{c}-\boldsymbol{r}_k|^2$$

is its variance (see [**6**]). Thus, although the applications in this paper are confined to geometry, the scope of possible applications is much larger.

The next result, a consequence of Theorem 1, provides a fundamental relation that is basic to all applications in this paper. Before stating it we establish a notation convention: in this paper we use $\sum_{k<i}$ as an abbreviation for a double sum

$$\sum_{i=2}^{n}\sum_{k=1}^{i-1}.$$

Theorem 2. *Let $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n$ be n points in m-space with weighted average $\boldsymbol{c}$. Then the following relation holds:*

$$\sum_{k<i} w_i w_k |\boldsymbol{r}_i - \boldsymbol{r}_k|^2 = W\sum_{k=1}^{n} w_k|\boldsymbol{z}-\boldsymbol{r}_k|^2 - W^2|\boldsymbol{c}-\boldsymbol{z}|^2, \qquad (4)$$

where $\boldsymbol{z}$ is arbitrary. In particular, when $\boldsymbol{z} = O$ this becomes

$$\sum_{k<i} w_i w_k |\boldsymbol{r}_i - \boldsymbol{r}_k|^2 = W\sum_{k=1}^{n} w_k|\boldsymbol{r}_k|^2 - W^2|\boldsymbol{c}|^2, \qquad (5)$$

and when $\boldsymbol{z} = \boldsymbol{c}$ it reduces to

$$\sum_{k<i} w_i w_k |\boldsymbol{r}_i - \boldsymbol{r}_k|^2 = W\sum_{k=1}^{n} w_k|\boldsymbol{c}-\boldsymbol{r}_k|^2. \qquad (6)$$

Proof. Taking $z = \boldsymbol{r}_i$ in (3), we obtain

$$\sum_{k=1}^{n} w_k |\boldsymbol{r}_i - \boldsymbol{r}_k|^2 = \sum_{k=1}^{n} w_k |\boldsymbol{c} - \boldsymbol{r}_k|^2 + W|\boldsymbol{c} - \boldsymbol{r}_i|^2$$

for $i = 1, 2, \ldots, n$, where, of course, the term with $k = i$ in the first sum is zero. Now multiply by w_i and sum on i to get

$$\sum_{i=1}^{n} \sum_{k=1}^{n} w_i w_k |\boldsymbol{r}_i - \boldsymbol{r}_k|^2 = W \sum_{k=1}^{n} w_k |\boldsymbol{c} - \boldsymbol{r}_k|^2 + W \sum_{i=1}^{n} w_i |\boldsymbol{c} - \boldsymbol{r}_i|^2$$
$$= 2W \sum_{k=1}^{n} w_k |\boldsymbol{c} - \boldsymbol{r}_k|^2.$$

This is equivalent to (6), because each term in the double sum appears twice. Now from (3) we have

$$W \sum_{k=1}^{n} w_k |\boldsymbol{c} - \boldsymbol{r}_k|^2 = W \sum_{k=1}^{n} w_k |z - \boldsymbol{r}_k|^2 - W^2 |\boldsymbol{c} - z|^2,$$

which, when substituted for the right-hand member of (6), gives (4). ■

3. EQUAL WEIGHTS. When all weights are equal, the weighted average $\boldsymbol{c}$ is called the *centroid* of the given set of points, and the foregoing relations connect sums of squares of distances among given points with squares of distances involving their centroid $\boldsymbol{c}$. If each $w_k = w$, say, then $W = nw$, and the common factor w cancels in each of the foregoing equations, which can now be written as indicated in equations (3′) through (6′).

From (3) we find that for any z in m-space,

$$\sum_{k=1}^{n} |z - \boldsymbol{r}_k|^2 = \sum_{k=1}^{n} |\boldsymbol{c} - \boldsymbol{r}_k|^2 + n|\boldsymbol{c} - z|^2. \tag{3′}$$

The special case with $\boldsymbol{c} = O$ was applied elsewhere in [**1**]–[**4**]. Specializing (4), (5), and (6) to equal weights we get:

$$\sum_{k<i} |\boldsymbol{r}_i - \boldsymbol{r}_k|^2 = n \sum_{k=1}^{n} |z - \boldsymbol{r}_k|^2 - n^2 |\boldsymbol{c} - z|^2 \textit{ for any } z \textit{ in } m\textit{-space}, \tag{4′}$$

$$\sum_{k<i} |\boldsymbol{r}_i - \boldsymbol{r}_k|^2 = n \sum_{k=1}^{n} |\boldsymbol{r}_k|^2 - n^2 |\boldsymbol{c}|^2, \tag{5′}$$

and

$$\sum_{k<i} |\boldsymbol{r}_i - \boldsymbol{r}_k|^2 = n \sum_{k=1}^{n} |\boldsymbol{c} - \boldsymbol{r}_k|^2, \tag{6′}$$

which gives an exact relation connecting the sum of the squares of all distances between arbitrary points $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n$ in m-space with the sum of squares of all their distances from the centroid.

For points in a plane, equations (5′) and (6′) can be found in Steiner [**9**, p. 108, (VI) and (VII)], and we thought their extensions to higher-dimensional space must surely be known. But our search of the literature did not uncover these exact formulas involving the centroid, even for three-dimensional space.

4. APPLICATIONS TO GEOMETRY.

The importance of the foregoing results is revealed by a multitude of applications.

Example 1 (Distances between arbitrary points). The exact relation in equation (5′) implies the inequality

$$\sum_{k=1}^{n} |\boldsymbol{r}_k|^2 \geq \frac{1}{n} \sum_{k<i} |\boldsymbol{r}_i - \boldsymbol{r}_k|^2, \tag{7}$$

with equality if and only if the centroid $\boldsymbol{c}$ is at the origin. This tells us that the sum of squares of distances from a given point O to n fixed points $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n$ is minimal when the point O is at their centroid, and this minimal value is the right-hand side of (7). This remarkable result was observed by Steiner for points in a plane ($m = 2$), but might not have been previously recorded for $m > 2$.

Example 2 (Points contained on a sphere in *m*-space). If $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n$, and $\boldsymbol{z}$ lie on a sphere of radius R with center at the origin, and if $\boldsymbol{c} = O$, equation (3′) reduces to

$$\sum_{k=1}^{n} |\boldsymbol{z} - \boldsymbol{r}_k|^2 = 2nR^2,$$

a generalization of the Pythagorean Theorem (which is the special case $n = 2$ when $\boldsymbol{r}_1 + \boldsymbol{r}_2 = O$).

Now consider the case when all points $\boldsymbol{r}_k$ lie inside or on a sphere of radius R with center at the origin. Then $|\boldsymbol{r}_k| < R$ if $\boldsymbol{r}_k$ is *inside* the sphere, and $|\boldsymbol{r}_k| = R$ if $\boldsymbol{r}_k$ is *on* the sphere, so the sum on the left of (7) has the upper bound

$$\sum_{k=1}^{n} |\boldsymbol{r}_k|^2 \leq nR^2,$$

with equality if and only if all $\boldsymbol{r}_k$ are on the sphere. Equation (5′) now gives us:

Theorem 3. *For any points $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n$ in m-space with centroid $\boldsymbol{c}$, all lying inside or on a sphere of radius R with center at the origin, the sum of the squares of all distances $|\boldsymbol{r}_i - \boldsymbol{r}_k|$ satisfies the inequality*

$$\sum_{k<i} |\boldsymbol{r}_i - \boldsymbol{r}_k|^2 \leq n^2(R^2 - |\boldsymbol{c}|^2), \tag{8}$$

with equality if and only if all points $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n$ are on the sphere, in which case

$$\sum_{k<i} |\boldsymbol{r}_i - \boldsymbol{r}_k|^2 = n^2(R^2 - |\boldsymbol{c}|^2). \tag{9}$$

This result provides a solution of Problem A4 of the 29th William Lowell Putnam Mathematical Competition [**7**], which asked to show that the sum of the squares of the $n(n-1)/2$ distances between any n distinct points on the surface of a unit sphere in 3-space is at most n^2. (See also [**5**] and 9.7 in [**8**].) The explicit appearance of the centroid in (9) gives a deeper understanding of the problem. Equation (9) gives an exact formula for the sum of the squares of the distances between any n points, no matter where they are located on a sphere in m-space, and it tells us that the sum reaches its maximum value n^2R^2 when the centroid is at the center of the sphere. This explains the surprising result that the maximum is independent of the dimensionality of the space. Any solution of the problem for n points lying on a circle is also a solution for n points lying on a sphere in m-space for all $m \geq 3$.

Example 3 (Sum of squares of edge lengths of a simplex). A simplex in m-space contains $m+1$ vertices $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_{m+1}$ (which always lie on the circumscribed sphere), and $\binom{m+1}{2} = (m+1)m/2$ edges joining these vertices. Then $|\boldsymbol{r}_i - \boldsymbol{r}_k|$ is the length of the edge joining $\boldsymbol{r}_i$ and $\boldsymbol{r}_k$, and $|\boldsymbol{c} - \boldsymbol{r}_k|$ is the distance from the centroid $\boldsymbol{c}$ to vertex $\boldsymbol{r}_k$. Equation (6′) asserts that the sum of the squares of the lengths of the edges of a simplex is exactly $m+1$ times the sum of the squares of the distances from the centroid to the vertices. Inequality (7) now implies the inequality

$$\sum_{k<i} |\boldsymbol{r}_i - \boldsymbol{r}_k|^2 \leq (m+1) \sum_{k=1}^{m+1} |\boldsymbol{r}_k|^2, \tag{10}$$

and tells us that the same sum is no more than $m+1$ times the sum of the squares of the distances from any point O to the vertices, with equality if and only if point O is the centroid.

In particular, (10) implies that in 2-space the sum of the squares of the lengths of the edges of any triangle is (a) three times the sum of the squares of the distances from the centroid to the vertices, and (b) no more than three times the sum of the squares of the distances from any point in the plane to the vertices. These properties of the triangle are known, but the authors were unable to locate the corresponding results in the literature for a tetrahedron or for any higher-dimensional simplex.

Example 4 (Sum of edge lengths of a simplex). The results in Example 3 for the sum of squares of the edge lengths of a simplex lead to upper bounds for the sum of the edge lengths themselves. The connection is provided by the Cauchy-Schwarz inequality, which states that

$$(\boldsymbol{u} \cdot \boldsymbol{v})^2 \leq |\boldsymbol{u}|^2 |\boldsymbol{v}|^2 \tag{11}$$

for any two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in a finite-dimensional Euclidean space, with equality if and only if one of the vectors is a scalar multiple of the other. We use (11) to prove the following lemma concerning the sum of edge lengths of any simplex.

Lemma. *The square of the sum of the edge lengths of any simplex in m-space is less than or equal to the number of edges times the sum of squares of the lengths of its edges, with equality if and only if the simplex is regular.*

Proof. Denote the edge lengths of a simplex in m-space by $d_1, d_2, \ldots, d_N$, where $N = m(m+1)/2$ is the number of edges, and consider the following two vectors in N-space:

$$\boldsymbol{u} = (1, 1, \ldots, 1), \qquad \boldsymbol{v} = (d_1, d_2, \ldots, d_N).$$

Then (11) gives us

$$\left(\sum_{k=1}^{N} d_k\right)^2 \le N \sum_{k=1}^{N} d_k^2, \tag{12}$$

with equality if and only if $d_1 = d_2 = \cdots = d_N$. ■

Because $|\boldsymbol{r}_i - \boldsymbol{r}_k|$ is the length of the edge joining $\boldsymbol{r}_i$ and $\boldsymbol{r}_k$, inequality (12) becomes

$$\left(\sum_{k<i} |\boldsymbol{r}_i - \boldsymbol{r}_k|\right)^2 \le \frac{m(m+1)}{2} \sum_{k<i} |\boldsymbol{r}_i - \boldsymbol{r}_k|^2, \tag{13}$$

with equality if and only if the simplex is regular.

In particular, when $m = 2$, this is a well-known inequality [**8**, p. 147]. It states that the square of the perimeter of any triangle is less than or equal to three times the sum of the squares of the lengths of its edges, with equality if and only if the triangle is equilateral. And its three-dimensional counterpart says that the square of the sum of all the edge lengths of a tetrahedron is less than or equal to six times the sum of the squares of the lengths of its edges, with equality if and only if the tetrahedron is regular.

Taking square roots in (13), we are led to the following basic inequality relating the sum of edge lengths of a simplex with the sum of squares of edge lengths:

$$\sum_{k<i} |\boldsymbol{r}_i - \boldsymbol{r}_k| \le \sqrt{\frac{m(m+1)}{2} \sum_{k<i} |\boldsymbol{r}_i - \boldsymbol{r}_k|^2}, \tag{14}$$

with equality if and only if the simplex is regular.

Using relations (6′) and (5′) for the sum of squares of edge lengths in (14), we discover corresponding results for the sum of edge lengths themselves, which we collect in the following theorem.

Theorem 4. *Let $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_{m+1}$ be the vertices of a simplex in m-space with centroid $\boldsymbol{c}$. Then the sum of edge lengths satisfies each of the following inequalities:*

$$\sum_{k<i} |\boldsymbol{r}_i - \boldsymbol{r}_k| \le (m+1)\sqrt{\frac{m}{2} \sum_{k=1}^{m+1} |\boldsymbol{c} - \boldsymbol{r}_k|^2}, \tag{15}$$

$$\sum_{k<i} |\boldsymbol{r}_i - \boldsymbol{r}_k| \le (m+1)\sqrt{\frac{m}{2} \sum_{k=1}^{m+1} |\boldsymbol{r}_k|^2 - (m+1)|\boldsymbol{c}|^2}, \tag{16}$$

with equality in each case if and only if the simplex is regular and $\boldsymbol{c}$ is at the origin. When $\boldsymbol{c} = O$ each of these reduces to

$$\sum_{k<i} |\boldsymbol{r}_i - \boldsymbol{r}_k| \le (m+1)\sqrt{\frac{m}{2} \sum_{k=1}^{m+1} |\boldsymbol{r}_k|^2}, \tag{17}$$

with equality if and only if the simplex is regular.

Now suppose that each vertex of the simplex lies inside or on a sphere of radius R with center at the origin. Then we can use inequality (8) from Theorem 3 (with $n = m + 1$) in the right-hand side of (14) to obtain

$$\sum_{k<i} |\boldsymbol{r}_i - \boldsymbol{r}_k| \le \sqrt{\frac{m(m+1)}{2}}(m+1)\sqrt{R^2 - |\boldsymbol{c}|^2}. \tag{18}$$

Equality holds in (18) if and only if the simplex is regular and has all its vertices on the sphere, in which case $\boldsymbol{c} = O$ and we get

$$\sum_{k<i} |\boldsymbol{r}_i - \boldsymbol{r}_k| = \sqrt{\frac{m(m+1)}{2}}(m+1)R \tag{19}$$

for the sum of edge lengths. This agrees with a result found by Chakerian and Klamkin [**5**], who showed that among all simplices inscribed in the unit sphere, the regular simplex has maximum total edge length. If d denotes the common value of the edge lengths, the left-hand side of (19) is $dm(m+1)/2$, and (19) implies that

$$d = \sqrt{\frac{2(m+1)}{m}}R,$$

the same result found in [**2**].

When $m = 2$, equation (19) gives $3\sqrt{3}R$ as the maximum perimeter of a triangle inscribed in a circle of radius R, and the maximum is achieved for an equilateral triangle. The corresponding maximum sum of edge lengths for a tetrahedron in 3-space is $4\sqrt{6}R$, and it is achieved for an inscribed regular tetrahedron.

5. COMPOSITE SYSTEMS. This section uses Theorem 1 to deduce a fundamental property of a concept we call the intrinsic second moment. If $w_1, w_2, \dots, w_n$ are n positive weights attached to n points $\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_n$ in m-space with weighted average position vector $\boldsymbol{c}$ defined by (1), we refer to the quantity

$$I = \sum_{k=1}^{n} w_k |\boldsymbol{r}_k - \boldsymbol{c}|^2 \tag{20}$$

as the *intrinsic second moment* of the system.

A system of n_1 points $\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_{n_1}$ in m-space with weights of sum W_1, average position vector $\boldsymbol{c}_1$, and intrinsic second moment I_1, taken together with a disjoint system of n_2 points $\boldsymbol{r}_{n_1+1}, \boldsymbol{r}_{n_1+2}, \dots, \boldsymbol{r}_{n_1+n_2}$ with weights of sum W_2, average position vector $\boldsymbol{c}_2$, and intrinsic second moment I_2, forms a composite system consisting of $n_1 + n_2$ points with average position vector $\boldsymbol{c}$ and intrinsic second moment I. The intrinsic second moments are related as follows:

Theorem 5. *The intrinsic second moment of a composite system is related to the intrinsic second moments of the two component systems by the equation*

$$I = I_1 + I_2 + \frac{W_1 W_2}{W_1 + W_2}|\boldsymbol{c}_1 - \boldsymbol{c}_2|^2. \tag{21}$$

The quantity $W_1 W_2/(W_1 + W_2)$ is called the *reduced weight* of the composite system (by analogy with reduced mass in physics). Equation (21) states that the intrinsic second moment of a composite body is equal to the sum of the intrinsic second moments of the component parts plus the reduced weight times the square of the distance between their average position vectors. For $m = 2$, this is a known result for the moment of inertia of a composite body.

Proof. The definition in (20) with $n = n_1 + n_2$ gives us

$$I = \sum_{(1)} w_k |\boldsymbol{r}_k - \boldsymbol{c}|^2 + \sum_{(2)} w_k |\boldsymbol{r}_k - \boldsymbol{c}|^2,$$

where $\sum_{(1)}$ signifies a sum taken over the n_1 points in one system and $\sum_{(2)}$ indicates summation over the n_2 points in the other. Applying Theorem 1 to each sum on the right with $\boldsymbol{z} = \boldsymbol{c}$, we obtain

$$\sum_{(1)} w_k |\boldsymbol{r}_k - \boldsymbol{c}|^2 = \sum_{(1)} w_k |\boldsymbol{r}_k - \boldsymbol{c_1}|^2 + W_1 |\boldsymbol{c_1} - \boldsymbol{c}|^2,$$

and there is a corresponding formula for $\sum_{(2)}$. Adding these equations, we learn that

$$I = I_1 + I_2 + W_1 |\boldsymbol{c_1} - \boldsymbol{c}|^2 + W_2 |\boldsymbol{c}_2 - \boldsymbol{c}|^2,$$

so to prove (21) it suffices to show that

$$W_1 |\boldsymbol{c_1} - \boldsymbol{c}|^2 + W_2 |\boldsymbol{c}_2 - \boldsymbol{c}|^2 = \frac{W_1 W_2}{W_1 + W_2} |\boldsymbol{c}_1 - \boldsymbol{c}_2|^2. \tag{22}$$

From (1) we have $(W_1 + W_2)\boldsymbol{c} = W_1 \boldsymbol{c}_1 + W_2 \boldsymbol{c}_2$, hence the three averages $\boldsymbol{c}$, $\boldsymbol{c_1}$, and $\boldsymbol{c}_2$ are collinear, and $W_1(\boldsymbol{c} - \boldsymbol{c_1}) = W_2(\boldsymbol{c}_2 - \boldsymbol{c})$, which implies that

$$W_1 |\boldsymbol{c} - \boldsymbol{c_1}| = W_2 |\boldsymbol{c}_2 - \boldsymbol{c}| \tag{23}$$

and

$$|\boldsymbol{c}_1 - \boldsymbol{c}| + |\boldsymbol{c}_2 - \boldsymbol{c}| = |\boldsymbol{c}_1 - \boldsymbol{c}_2|. \tag{24}$$

Solving for $|\boldsymbol{c}_2 - \boldsymbol{c}|$ and substituting in (23), we find that

$$(W_1 + W_2)|\boldsymbol{c} - \boldsymbol{c_1}| = W_2 |\boldsymbol{c}_1 - \boldsymbol{c}_2|,$$

from which we conclude that

$$W_1 |\boldsymbol{c} - \boldsymbol{c_1}| = \frac{W_1 W_2}{W_1 + W_2} |\boldsymbol{c}_1 - \boldsymbol{c}_2|. \tag{25}$$

Now return to the left-hand member of (22) and rewrite it as follows:

$$(W_1 |\boldsymbol{c}_1 - \boldsymbol{c}|)|\boldsymbol{c}_1 - \boldsymbol{c}| + (W_2 |\boldsymbol{c}_2 - \boldsymbol{c}|)|\boldsymbol{c}_2 - \boldsymbol{c}|.$$

By (23) we can replace $W_2 |\boldsymbol{c}_2 - \boldsymbol{c}|$ in the second term with $W_1 |\boldsymbol{c}_1 - \boldsymbol{c}|$, and the sum becomes

$$(W_1 |\boldsymbol{c}_1 - \boldsymbol{c}|)(|\boldsymbol{c}_1 - \boldsymbol{c}| + |\boldsymbol{c}_2 - \boldsymbol{c}|) = (W_1 |\boldsymbol{c}_1 - \boldsymbol{c}|)(|\boldsymbol{c}_1 - \boldsymbol{c}_2|) = \frac{W_1 W_2}{W_1 + W_2} |\boldsymbol{c}_1 - \boldsymbol{c}_2|^2,$$

where we have used (24) and (25). This proves (22) and hence (21). ■

Relations not requiring the average position vector. Although the intrinsic second moment is defined by (20) in terms of the average position vector $\boldsymbol{c}$, it can also be expressed in a form that does not involve $\boldsymbol{c}$, but only distances between points of the body. In fact, by using (6) of Theorem 2 in (20) we arrive at the expression

$$I = \frac{1}{W} \sum_{k<i} w_i w_k |\boldsymbol{r}_i - \boldsymbol{r}_k|^2, \tag{26}$$

and similar formulas hold for I_1 and I_2. Now multiply each side of (21) by $W = W_1 + W_2$ and rewrite it in the alternate form

$$\sum_{k<i} w_i w_k |\boldsymbol{r}_i - \boldsymbol{r}_k|^2 = \frac{W}{W_1} \sum_{(1)} + \frac{W}{W_2} \sum_{(2)} + W_1 W_2 |\boldsymbol{c}_1 - \boldsymbol{c}_2|^2, \tag{27}$$

where the sum on the left is extended over all points of the composite system, while $\sum_{(1)}$ and $\sum_{(2)}$ are the corresponding sums with the same summand $w_i w_k |\boldsymbol{r}_i - \boldsymbol{r}_k|^2$, but extended over the points of the respective component system. But we also have

$$\sum_{k<i} w_i w_k |\boldsymbol{r}_i - \boldsymbol{r}_k|^2 = \sum_{(1)} + \sum_{(2)} + \sum_{(1,2)},$$

where $\sum_{(1,2)}$ signifies summation with the same summand over all vectors $\boldsymbol{r}_i - \boldsymbol{r}_k$ joining points from *different* component systems. Use this to replace the left-hand side of (27), and solve for $\sum_{(1,2)}$ to obtain

$$\sum_{(1,2)} = \frac{W_2}{W_1} \sum_{(1)} + \frac{W_1}{W_2} \sum_{(2)} + W_1 W_2 |\boldsymbol{c}_1 - \boldsymbol{c}_2|^2. \tag{28}$$

In other words, the double sum $\sum_{k<i} w_i w_k |\boldsymbol{r}_i - \boldsymbol{r}_k|^2$ extended over segments joining all points $\boldsymbol{r}_i$ and $\boldsymbol{r}_k$ taken from two different component systems is a linear combination of sums extended over the separate systems plus the product of the weights times the square of the distance between the average position vectors of the two systems. This equation can be used to calculate the distance $|\boldsymbol{c}_1 - \boldsymbol{c}_2|$ without knowing $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ explicitly. And, if (28) is combined with (25), we can also compute the distance $|\boldsymbol{c} - \boldsymbol{c}_1|$ without knowing $\boldsymbol{c}$ or $\boldsymbol{c}_1$. In this case the formula for $|\boldsymbol{c} - \boldsymbol{c}_1|$ becomes

$$|\boldsymbol{c} - \boldsymbol{c}_1|^2 = \frac{1}{W^2 W_1^2} \left(W_1 W_2 \sum_{(1,2)} - W_2^2 \sum_{(1)} - W_1^2 \sum_{(2)} \right), \tag{29}$$

with summand $w_i w_k |\boldsymbol{r}_i - \boldsymbol{r}_k|^2$ in each sum on the right.

6. EQUAL WEIGHTS: APPLICATIONS TO GEOMETRY. This section considers composite systems having equal weights, with the total weight of each system replaced by the corresponding number of points n_1, n_2, and $n = n_1 + n_2$. Formulas (27) and (29) of section 5 have interesting applications to geometry.

Example 5 (General simplex). An m-dimensional simplex with $m + 1$ vertices can be regarded as a composite system made up of two parts, one part having exactly one vertex, which is also its centroid $\boldsymbol{c}_1$, and the other part consisting of the remaining m

vertices. Using (29) to determine the distance from this one vertex $\boldsymbol{c}_1$ to the centroid $\boldsymbol{c}$ of the entire simplex, we find that

$$|\boldsymbol{c} - \boldsymbol{c}_1| = \frac{1}{m+1}\sqrt{m\sum_{(1,2)} - \sum_{(2)}},$$

a result consistent with (3′) when $n = m + 1$ and $z = \boldsymbol{c}_1$. In this case, $\sum_{(1,2)}$ is the sum of squares of distances from $\boldsymbol{c}_1$ to all adjacent vertices, $\sum_{(1)}$ is zero, and $\sum_{(2)}$ is the sum of squares of all the remaining edges. For example, for a triangle ($m = 2$) with edges of lengths a_1, a_2, and a_3, where a_1 and a_2 are the lengths of the edges adjacent to $\boldsymbol{c}_1$, we have

$$|\boldsymbol{c} - \boldsymbol{c}_1| = \frac{1}{3}\sqrt{2(a_1^2 + a_2^2) - a_3^2}.$$

The corresponding formula for a tetrahedron ($m = 3$) is

$$|\boldsymbol{c} - \boldsymbol{c}_1| = \frac{1}{4}\sqrt{3(a_1^2 + a_2^2 + a_3^3) - (a_4^2 + a_5^2 + a_6^2)},$$

where a_1, a_2, and a_3 are the lengths of the edges adjacent to $\boldsymbol{c}_1$ and a_4, a_5, and a_6 are the lengths of the remaining edges.

Example 6 (Quadrilaterals). Let $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3$, and $\boldsymbol{r}_4$ be four distinct points in m-space. Take $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ as one system, and $\boldsymbol{r}_3$ and $\boldsymbol{r}_4$ as another. Use equal weights, let $\boldsymbol{c}_1$ be the midpoint of the segment joining $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, and let $\boldsymbol{c}_2$ be the midpoint of the segment joining $\boldsymbol{r}_3$ and $\boldsymbol{r}_4$. Equation (28) now becomes

$$\begin{aligned} &|\boldsymbol{r}_1 - \boldsymbol{r}_3|^2 + |\boldsymbol{r}_1 - \boldsymbol{r}_4|^2 + |\boldsymbol{r}_2 - \boldsymbol{r}_3|^2 + |\boldsymbol{r}_2 - \boldsymbol{r}_4|^2 \\ &\quad = |\boldsymbol{r}_1 - \boldsymbol{r}_2|^2 + |\boldsymbol{r}_3 - \boldsymbol{r}_4|^2 + 4|\boldsymbol{c}_1 - \boldsymbol{c}_2|^2. \end{aligned} \tag{30}$$

The points $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \boldsymbol{r}_4$ can be regarded as the vertices of a quadrilateral, with the sum on the left being the sum of the squares of the lengths of the four edges, and the first two terms on the right being the sum of the squares of the two diagonals. Equation (30) states that the sum of the squares of the edges exceeds the sum of the squares of the diagonals by four times the square of the distance between the midpoints of the diagonals. This extends a well-known planar result to m-space. It also shows that the sum of the squares of the diagonals is at most the sum of the squares of its four edges, with equality if and only if $\boldsymbol{c}_1 = \boldsymbol{c}_2$. The latter occurs only when the diagonals bisect each other, that is, when the quadrilateral is a parallelogram lying in a plane. The next example gives a corresponding result for hexagons.

Example 7 (Hexagons). Let $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_6$ be six distinct points in m-space joined in sequence to form a hexagon. Vertices $\boldsymbol{r}_1, \boldsymbol{r}_3$, and $\boldsymbol{r}_5$ form one triangle with centroid $\boldsymbol{c}_1$, and vertices $\boldsymbol{r}_2, \boldsymbol{r}_4$, and $\boldsymbol{r}_6$ form another triangle with centroid $\boldsymbol{c}_2$. In general, these triangles are in different planes, and they may or may not intersect. The hexagon has six minor diagonals that form the edges of the two triangles, plus three major diagonals joining $\boldsymbol{r}_1$ and $\boldsymbol{r}_4$, $\boldsymbol{r}_2$ and $\boldsymbol{r}_5$, and $\boldsymbol{r}_3$ and $\boldsymbol{r}_6$, respectively. Take equal weights and use the abbreviation r_{ij}^2 for $|\boldsymbol{r}_i - \boldsymbol{r}_j|^2$, and (28) becomes

$$\begin{aligned} &(r_{12}^2 + r_{23}^2 + r_{34}^2 + r_{45}^2 + r_{56}^2 + r_{61}^2) + (r_{14}^2 + r_{25}^2 + r_{36}^2) \\ &= (r_{13}^2 + r_{35}^2 + r_{51}^2 + r_{24}^2 + r_{46}^2 + r_{62}^2) + 9|\boldsymbol{c}_1 - \boldsymbol{c}_2|^2. \end{aligned}$$

This states that the sum of squares of the six edges and the three major diagonals of a hexagon exceeds the sum of the squares of its six minor diagonals by nine times the square of the distance between the centroids of the two triangles. Equality holds if and only if the centroids of the two triangles coincide, although the hexagon need not in that case be flat.

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