Isoperimetric and Isoparametric Problems

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1. INTRODUCTION. Two incongruent solids with remarkable properties are shown in Figure 1. One is a slice of a solid hemispherical shell with inner radius r and outer radius R cut by a plane parallel to the equator and at distance $h < r$ from the equator. The other is a cylindrical shell with the same radii and altitude h. The surface of each solid consists of four components: an upper circular ring, a lower circular ring, an outer lateral surface, and an inner lateral surface. The two solids share the following properties:

(a) *The solids have equal volumes.*

(b) *The solids have equal total surface area.*

(c) *Every plane parallel to the equator cuts both solids in cross sections of equal area.*

(d) *The two inner lateral surface areas are equal.*

(e) *The two outer lateral surface areas are equal.*

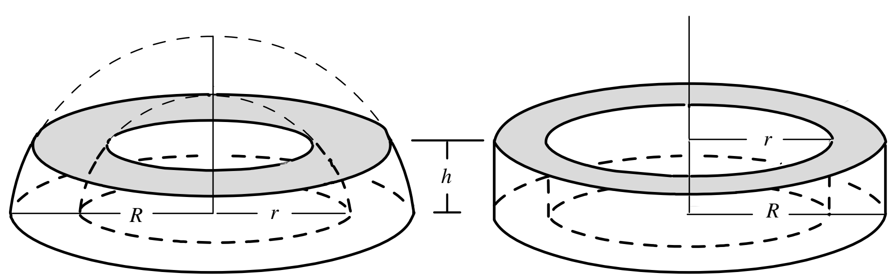


Figure 1. Incongruent solids sharing the five properties (a) through (e).

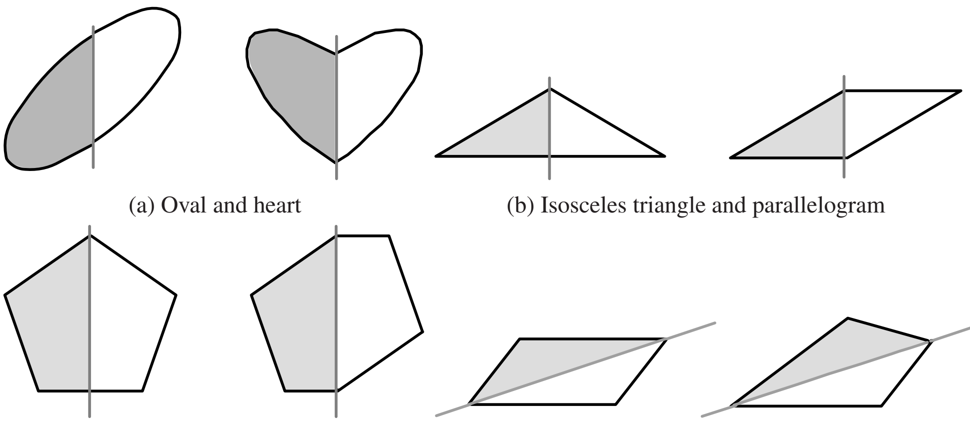
These two solids were introduced in [**1**], where it is shown that the five properties (a)–(e) are also shared by an entire family of incongruent solids, each of which has polygonal rings as cross sections, rings formed by similar polygons circumscribing the inner and outer circular cross sections of a spherical shell. These solids, in turn, are part of a more general family with polygonal rings as cross sections (described in section 8) that is even more remarkable because it satisfies these five properties and a sixth property not shared by the two solids in Figure 1:

(f) *Every plane parallel to the equator cuts both solids in cross-sectional rings whose inner perimeters are equal and whose outer perimeters are equal.*

The last property implies that the two cross-sectional rings also have the same total perimeter. This observation motivated the present paper, which is concerned with plane regions having equal areas and equal perimeters. Because two global parameters (area and perimeter) are to be equal, we call such regions *isoparametric*. The first problem we posed was:

How can we construct incongruent isoparametric plane regions?

With no further restrictions on the regions, it is easy to produce such examples at will, as shown by the regions in Figure 2. A chord divides each region into two pieces. One piece is flipped to produce an incongruent isoparametric region. It is clear that infinitely many such regions can be produced in this way by cutting and flipping a piece from any given region, unless it is a circular disk, in which case flipping one of the pieces results in a congruent disk.



(a) Oval and heart (b) Isosceles triangle and parallelogram

(c) Regular pentagon and hexagon (d) Parallelogram and kite

Figure 2. Incongruent isoparametric plane regions formed by cutting and flipping.

Traditional isoperimetric problems compare different plane regions having equal perimeters and ask for the region of maximal area. It is known [**3**], [**5**] that among all regions with a given perimeter, the circle encloses the largest area. This follows from the *isoperimetric inequality*,

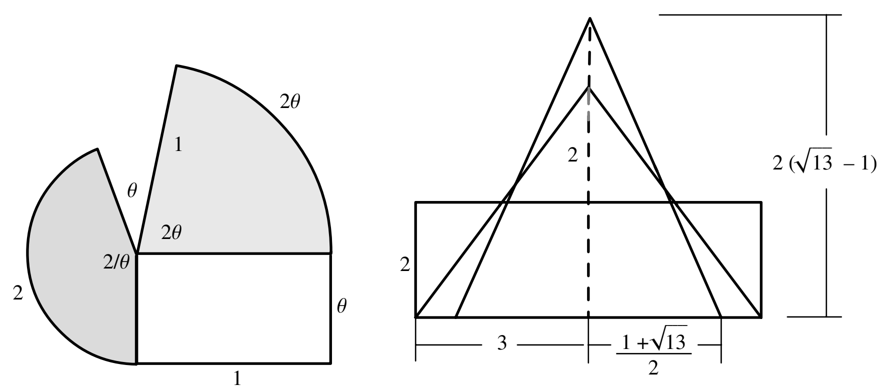
$$\frac{p^2}{4s} \geq \pi, \tag{1}$$

which relates the perimeter p and area s of any planar region bounded by a simple closed curve. Equality holds in (1) only for the circle.

Isoperimetric problems have been a source of important mathematical ideas and techniques since classical antiquity. A result arising from mythology is *Dido's problem*: in the half-plane bounded by a given line, find a curvilinear arc of prescribed length with its extremities on the line and enclosing the maximum area. The solution, a semicircle whose diameter is on the given line, is obtained by reflecting the curve in the line and invoking the isoperimetric property of the circle. Archimedes treats a three-dimensional analog in Proposition 9 of his *Sphere and Cylinder II*, which states that of all spherical segments having equal spherical surface area, the hemisphere has the greatest volume. Today, isoperimetric problems and their extensions are alive and well. They continue to nourish mathematical imagination, as evidenced by a recent proof of the double bubble conjecture [**4**]. Interesting historical perspective on isoperimetric problems is given in [**3**], which also describes the relation to a host of other maximum-minimum problems dealt with by a method called the *calculus of variations*.

This paper treats a different type of problem: *find incongruent plane regions that have equal perimeters and equal areas*. Hence the new name: *isoparametric problem*. The problem becomes more interesting, and more difficult, if we seek incongruent isoparametric regions *of specified shapes*. It cannot be solved if one of the regions is a circular disk because of the isoperimetric inequality. Also, it cannot be solved for two regular polygons with different numbers of sides, although the reason for this may not be obvious (see section 2).

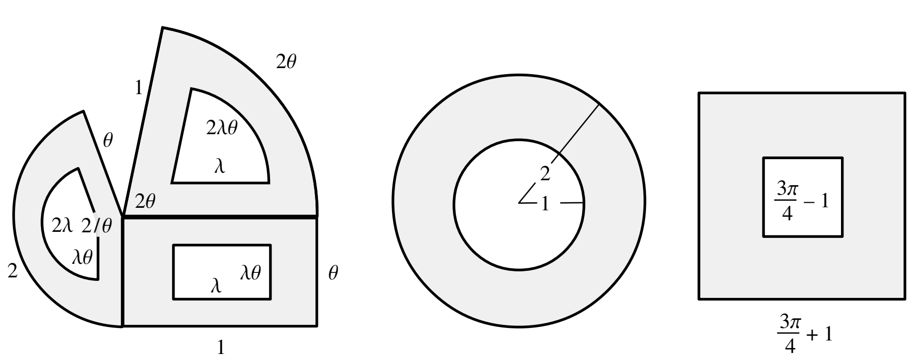
The authors were pleasantly surprised to discover numerous cases where it can be solved. Figure 3a shows three incongruent isoparametric regions: two circular sectors and a rectangle, each with area θ and perimeter $2 + 2\theta$. Because θ is arbitrary, this provides an infinite family of examples. When $\theta = 1$ the rectangle is a square and the two sectors are congruent. Figure 3b shows three more examples: two isosceles triangles and a rectangle, each with area 12 and perimeter 16. Such examples show that the general problem of finding incongruent isoparametric regions of specified shapes opens a door to many possibilities worth exploring. Section 3 gives a systematic treatment. Section 4 treats the same type of problem for rings bounded by two similar closed curves. Introducing "holes" makes the problem more interesting and allows more possibilities.



(a) Two circular sectors and a rectangle (b) Two isosceles triangles and a rectangle

Figure 3. Examples of incongruent isoparametric regions.

For example, Figure 4a shows three such rings formed from the sectors and rectangle in Figure 3a. The holes are obtained by shrinking each figure by the same size factor $\lambda (< 1)$. For any size factor λ, each of the three rings has area $(1 - \lambda^2)\theta$ and total perimeter $(1 + \lambda)(2 + \theta)$, so they are isoparametric. In Figure 4b a circular ring and



(a) Sectorial rings and rectangular ring (b) Circular ring and square ring

Figure 4. Examples of incongruent isoparametric rings.

a square ring have the same area 3π and the same total perimeter 6π. For the circular ring the size factor λ (ratio of inner to outer radius) is $1/2$. But if $\lambda < 1/9$, it turns out that there is no isoparametric square ring. Sections 5 and 6 explain why there is a restriction on λ in Figure 4b but not in Figure 4a. Section 7 discusses isoparametric rings that have property (f) mentioned earlier.

2. CONTOUR RATIOS. In this paper, a contour is any plane region that has associated with it a perimeter p and an area s. For a given region, the ratio $Q = 4\pi s/p^2$ is called the isoperimetric quotient. The isoperimetric inequality states that $Q \leq 1$ for regions bounded by simple closed curves, with $Q = 1$ only for the circle. Some properties of Q are given in [**5**].

For our purposes, it is more useful to study the quotient

$$\kappa = \frac{p^2}{4s}, \tag{2}$$

which we call the *contour ratio*. It has the pleasant feature that $\kappa = \pi$ for any circle and $\kappa = 4$ for any square. *Isoparametric contours have the same contour ratio.* A regular n-gon has contour ratio $\kappa = n\tan(\pi/n)$, a decreasing function of n that approaches π as $n \to \infty$. That is why regular polygons with different numbers of sides cannot be isoparametric.

For all contours with $s = 1$, the contour ratio is the square of the semiperimeter. Qualitatively, the contour ratio indicates the dominance of the semiperimeter over the square root of the area.

Similar contours have the same contour ratio because the scaling factor cancels in (2). Figure 5 shows various shapes arranged by contour ratios. The size of a region plays no role. All circles are located at π, and all squares at 4.

Figure 5 also provides a spectrum of contour ratios for families of various shapes. Regular polygons serve as discrete bench marks. For example, an equilateral triangle has contour ratio $3\sqrt{3} = 5.1961\ldots$. All other triangles have larger contour ratios, so their images are distributed continuously to the right of the equilateral triangle. The square, another bench mark, has the smallest contour ratio of all quadrilaterals. More generally, for each fixed n the images of all n-gons lie to the right of the regular n-gon, which has the smallest contour ratio among all n-gons.

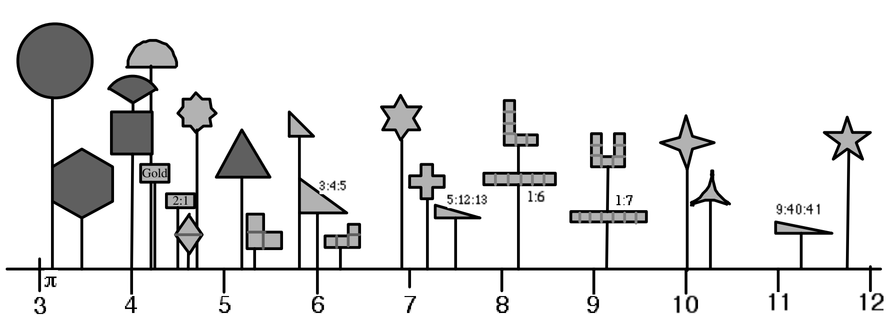


Figure 5. A spectrum of contour ratios of various shapes. Relative sizes are irrelevant.

An isosceles right triangle is also a bench mark. Its contour ratio is $3 + 2\sqrt{2} = 5.8284\ldots$, which is the smallest κ that occurs among all right triangles. In particular, any right triangle with integer sides (a Pythagorean triangle) has a larger κ. The $3:4:5$ triangle has $\kappa = 6$, and the $119:120:169$ triangle, which is nearly isosceles, has $\kappa = 5.8285\ldots$. We offer as a challenge to the reader to show that Pythagorean right triangles exist with κ arbitrarily close to $3 + 2\sqrt{2}$; thus, no Pythagorean triangle exists with smallest κ.

On the other hand, two regions can have the same contour ratio even though their shapes are quite different. In Figure 3a the rectangle has a different shape than the two sectors, but each has area θ and perimeter $2\theta + 2$, so their contour ratios are equal. The pentagram in Figure 5 has contour ratio $10\sqrt{3-\phi} = 11.7557\ldots$, where $\phi = (\sqrt{5}+1)/2$ is the golden ratio. A special long thin rectangle of some base $b \approx 9.6521$ and height 1 has the same contour ratio, even though its shape has no resemblance to a pentagram. Although the perimeter and area of this rectangle are not equal to those of the pentagram, there is a similar rectangle with exactly the same area and the same perimeter as the pentagram, as revealed by the following theorem.

Theorem 1 (Isoparametric Contour Theorem). *Two contours can be scaled to become isoparametric if and only if they have the same contour ratio.*

Proof. If two contours can be scaled to become isoparametric, then the scaled contours have the same contour ratio, hence so do the original contours because κ is invariant under scaling. Conversely, assume that contours 1 and 2 have equal contour ratios, $p_1^2/(4s_1) = p_2^2/(4s_2)$. Then $s_2/s_1 = (p_2/p_1)^2$. If we scale contour 1 by the scaling factor $t = p_2/p_1$ we obtain a similar contour with perimeter $tp_1 = p_2$ and area $t^2 s_1 = s_2$. This scaled copy of contour 1 is isoparametric to contour 2. In the same way, if we scale contour 2 by the scaling factor p_1/p_2 the scaled copy will be isoparametric to contour 1. ■

In terms of the traditional isoperimetric quotient, Theorem 1 states that two regions with the same isoperimetric quotient can be scaled to have equal perimeters and equal areas.

We call two contours *parametrically similar* if they have the same contour ratio. Isoparametric contours are always parametrically similar, whereas parametrically similar contours can be scaled to become isoparametric. Stated another way, isoparametric contours scaled differently are parametrically similar. A pentagram and the special rectangle of base $b \approx 9.6521$ and altitude 1 are parametrically similar but not isoparametric.

For later reference, we find the contour ratios of some specific shapes, with some simple consequences.

Example 1 (Polygons circumscribing a circle). Consider a polygon with perimeter p and area s that circumscribes a given circle of radius r. The polygon need not be regular. Then $s = rp/2$, so the contour ratio of the polygon is

$$\kappa_{\text{poly}} = \frac{p^2}{4s} = \frac{p}{2r}. \tag{3}$$

Note that this is the ratio of the perimeter to the diameter of the inscribed circle (just as π is the ratio of the circumference of a circle to its diameter). For polygons with n sides, the minimum value of κ_{poly} occurs when p is minimal, which means when the polygon is regular. In this case $\kappa = n\tan(\pi/n)$, a bench mark for all n-gons.

The perimeter and area of any circumscribing polygon can, in turn, be calculated in terms of the contour ratio. Using (3) we find that

$$p = 2r\kappa_{\text{poly}}, \qquad s = r^2\kappa_{\text{poly}}. \tag{4}$$

These generalize the classical formulas $p = 2r\pi$ and $s = r^2\pi$ for the circumference and area of the circumscribed circle. From the formulas in (4) we conclude:

Polygons with equal contour ratios that circumscribe the same circle are isoparametric.

Polygons with equal perimeters that circumscribe the same circle are isoparametric.

Polygons with equal areas that circumscribe the same circle are isoparametric.

Figure 6 shows an equilateral triangle and a rhombus that circumscribe the same unit circle. Each has perimeter $6\sqrt{3}$ and area $3\sqrt{3}$. A given circle can be circumscribed by a regular n-gon and by a regular m-gon. If $m \neq n$, these regular polygons necessarily have different perimeters, different areas, and different contour ratios. But an n-gon can be isoparametric to an m-gon if the one with the larger number of sides is not regular, as illustrated by the example in Figure 6.

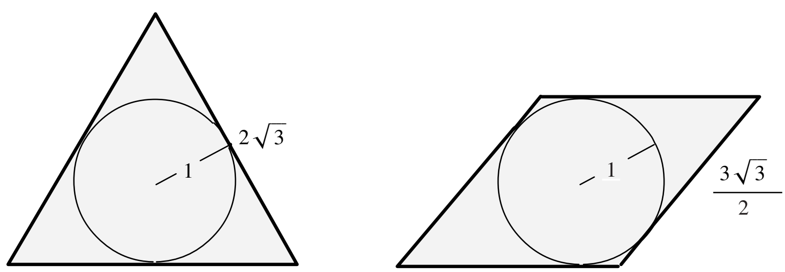


Figure 6. An equilateral triangle and an isoparametric rhombus circumscribing the same circle.

Polygons that circumscribe circles have a number of remarkable properties, some of which are alluded to in Example 1. A more comprehensive list of such properties is given in [**2**].

Example 2 (Rectangle). A rectangle of base b and altitude a has area ab and perimeter $2(a+b)$, so its contour ratio is

$$\kappa_{\text{rect}} = \frac{(a+b)^2}{ab} = 2 + \frac{a}{b} + \frac{b}{a} = 2 + \gamma + \frac{1}{\gamma}, \tag{5}$$

where γ is the ratio of the two edges. The minimum occurs when $\gamma = 1$, which gives a square with $\kappa = 4$, a bench mark for all rectangles.

From (5) we find that two rectangles with edge ratios γ_1 and γ_2 have the same contour ratio if and only if $\gamma_1 = \gamma_2$ or $\gamma_1\gamma_2 = 1$. In both cases the rectangles are similar. Therefore, *dissimilar rectangles cannot be isoparametric*.

Example 3 (Circular sector). A circular sector of unit radius subtending an angle of 2θ radians has area θ and perimeter $2 + 2\theta$. Therefore its contour ratio is

$$\kappa_{\text{sect}} = \frac{(2+2\theta)^2}{4\theta} = 2 + \theta + \frac{1}{\theta}. \tag{6}$$

It is both surprising and remarkable that this has the same form as (5), with γ replaced by θ. As illustrated in Figure 3a, each sector is isoparametric to a rectangle. Again, the minimum is $\kappa = 4$, the contour ratio of a square, and it occurs when $\theta = 1$. A circular sector subtending an angle of 2 radians appears as a bench mark in Figure 5. Sectors subtending angles greater than or less than 2 radians have contour ratio greater than 4.

Example 4 (Isosceles triangle). An isosceles triangle with equal legs d and vertex angle 2θ has area $s = d^2 \sin\theta \cos\theta$ and perimeter $p = 2d + 2d\sin\theta$, so its contour ratio is

$$\kappa_{\text{isos}} = \frac{4d^2(1+\sin\theta)^2}{4d^2\sin\theta\cos\theta} = \frac{1}{\cos\theta}\left(2 + \sin\theta + \frac{1}{\sin\theta}\right). \tag{7}$$

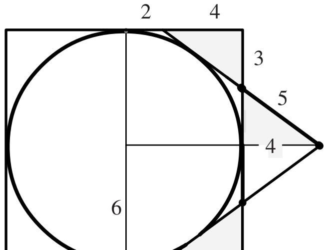
As expected, the right-hand side has its minimum value when $\theta = \pi/6$, giving $\kappa = 3\sqrt{3}$ for an equilateral triangle, a bench mark for all isosceles triangles.

3. ISOPARAMETRIC CONTOURS OF DIFFERENT SHAPES. This section solves some special cases of the following type of problem:

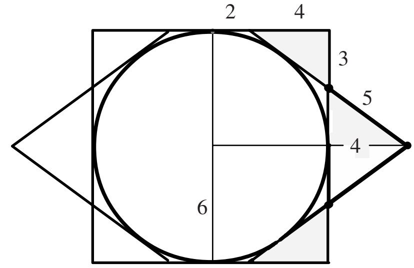
Isoparametric contour problem. *Given a contour of specified shape (such as a rectangle), under what conditions can we find an isoparametric contour of another specified shape (such as an isosceles triangle)?*

A necessary condition that the specified shapes be isoparametric is that they have the same contour ratio. If the contour ratios are equal the shapes are parametrically similar, and they can be scaled to become isoparametric.

Example 5 (Square, pentagon, and hexagon). Given a square, whose contour ratio is 4, we wish to find an isoparametric pentagon, which necessarily has contour ratio 4. A general pentagon involves many parameters (such as angles and lengths of edges). We can restrict some of them and still satisfy the requirement that the pentagon have contour ratio 4. It is easier to find a pentagon that circumscribes the same circle as the given square and has the same perimeter. (A regular pentagon will not do because it has contour ratio smaller than 4.) Figure 7a shows an example that works. A circle of radius 6 is circumscribed by a square of side-length 12. Two Pythagorean 3 : 4 : 5 triangles are cut off at two corners of the square by tangents to the circle and flipped to form two edges of a circumscribing pentagon. Because the square and pentagon have equal areas, they are isoparametric. In fact the circumscribing pentagon has two edges of length 10, two of length 8, and one of length 12, so its perimeter is 48, the same as that of the square. Both the square and pentagon have area 144.



(a) Isoparametric square and pentagon



(b) Isoparametric square and hexagon

Figure 7. Isoparametric square, pentagon, and hexagon circumscribing the same circle.

If the other two corners of the square are cut off in a similar manner, we find an example of a circumscribing hexagon, shown in Figure 7b, that is isoparametric to both the square and pentagon.

Example 6 (General contour and rectangle). Assume that we are given a general contour with contour ratio κ, and that we seek an isoparametric rectangle. If $\kappa < 4$ there is no rectangle isoparametric to the given contour because $\kappa_{\mathrm{rect}} \geq 4$ for every rectangle. So for this problem, $\kappa \geq 4$ is a *constraint* on the general contour. A necessary condition that they be isoparametric is that $\kappa_{\mathrm{rect}} = \kappa$, and from (5) we find that $\gamma^2 + (2-\kappa)\gamma + 1 = 0$, where γ is the side ratio of the rectangle. For given $\kappa \geq 4$ this quadratic equation for γ has positive roots given by

$$\gamma = \frac{1}{2}(\kappa - 2) \pm \frac{1}{2}\sqrt{\kappa(\kappa-4)} = \frac{1}{4}(\sqrt{\kappa} \pm \sqrt{\kappa - 4})^2. \tag{8}$$

The product of the roots is 1, so the roots are reciprocals. If $\kappa = 4$, then $\gamma = 1$ and the rectangle is a square. If $\kappa > 4$ there are two distinct roots, γ and $1/\gamma$, but geometrically they are obtained by interchanging the base and altitude of the same rectangle. The rectangle is parametrically similar to the given contour. By Theorem 1 the latter can be scaled to become isoparametric to the former.

There is an equivalent way to treat this problem. Suppose that the rectangle has base b and altitude a, and that the given contour has area s and perimeter p. There is no loss of generality if we take $p = 2$, which makes $\kappa = 1/s$. Equating perimeters and areas we find $a + b = 1$ and $ab = a(1-a) = s = 1/\kappa$, so

$$a(1-a) = \frac{1}{\kappa}. \tag{9}$$

Equation (9) is quadratic in a with two roots $(1 \pm \sqrt{1 - 4/\kappa})/2$, whose sum is 1. The constraint $\kappa \geq 4$ ensures that both roots are positive. If we take $a \leq b$, then a is given by

$$a = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4}{\kappa}} = \frac{1}{2} - \frac{1}{2}\sqrt{1-4s}, \tag{10}$$

and $b = 1 - a$. We will use this relation in Example 9.

Example 7 (General contour and circular sector). We take any contour with contour ratio κ and try to find an isoparametric circular sector of unit radius. Denote the central angle of the sector, measured in radians, by 2θ. If $\kappa < 4$ there is no sector isoparametric to the given contour because $\kappa_{\mathrm{sect}} \geq 4$ for every sector. Therefore, as in Example 6, $\kappa \geq 4$ is a constraint on the general contour. A necessary condition that the two be isoparametric is that $\kappa_{\mathrm{sect}} = \kappa$, hence by (6) we find that $\theta^2 + (2-\kappa)\theta + 1 = 0$, the same quadratic equation satisfied by γ in Example 6, whose roots are given by (8). We know from Example 3 that the sector is isoparametric to a rectangle with edges 1 and θ, so this problem is equivalent to Example 6. For each $\kappa \geq 4$, the quadratic has two positive roots that are reciprocals. If $\kappa = 4$, then $\theta = 1$ and there is one circular sector of unit radius subtending an angle of 2 radians. It is isoparametric to the unit square and can be scaled to become isoparametric to any contour with $\kappa = 4$.

If $\kappa > 4$, there are two dissimilar circular sectors of unit radius with the same κ; one subtends an angle of 2θ radians, the other an angle of $2/\theta$ radians. The two sectors are parametrically similar, but (except for the case $\theta = 1$) they are not isoparametric. But if we scale the second sector by a factor θ we obtain a similar sector that is isoparametric to the first sector and also to any given contour of contour ratio κ.

In particular, if the given contour is a rectangle with edges 1 and θ, there are two sectors isoparametric to the rectangle. They are shown in Figure 3a. Because having contour ratio 4 is a common bench mark for both sectors and rectangles, there are no restrictions on the parameter θ that defines the sector or the rectangle.

Example 8 (General contour and isosceles triangle). Here we start with a contour of contour ratio κ and seek an isoparametric isosceles triangle with parameter $t = \sin\theta$, where θ is half the vertex angle. Because $\kappa_{\text{isos}} \geq 3\sqrt{3}$ this problem places the constraint $\kappa \geq 3\sqrt{3}$ on the general contour. A necessary condition that the given contour and a specific isosceles triangle be isoparametric is that $\kappa_{\text{isos}} = \kappa$. Use (7) for κ_{isos} with $\sin\theta$ replaced by t to obtain

$$\kappa = \frac{1}{\sqrt{1-t^2}} \frac{(1+t)^2}{t} = \sqrt{\frac{1+t}{1-t}} \frac{1+t}{t}.$$

Now square both sides and rearrange terms to get

$$(1+\kappa^2)t^3 + (3-\kappa^2)t^2 + 3t + 1 = 0,$$

a cubic equation for t in terms of κ. When $t = 0$, the cubic polynomial on the left has the value 1, and when $t = -1$, it has the value $-2\kappa^2$, so it always has a root in the interval $(-1, 0)$. A negative root t does not correspond to a possible vertex angle 2θ, so we ignore it.

When $\kappa = 3\sqrt{3}$ the cubic equation becomes $(2t-1)^2(7t+1) = 0$, which has a double root $t = 1/2$ (and a negative root). The root $t = 1/2$ corresponds to $\theta = \pi/6$, which makes the triangle equilateral. When $\kappa > 3\sqrt{3}$, it can be shown that the cubic has two positive roots in the interval $(0, 1)$.

For example, when $\kappa = 3 + 2\sqrt{2}$ the equation becomes

$$\left(t - \frac{1}{2}\sqrt{2}\right)\left[(18+12\sqrt{2})t^2 - (2+3\sqrt{2})t - \sqrt{2}\right] = 0.$$

This cubic has a root at $t = \sqrt{2}/2$, corresponding to $2\theta = \pi/2$, which gives an isosceles right triangle. The quadratic factor has only one positive root $t = 0.3093\ldots$, corresponding to another isosceles triangle with a vertex angle of approximately 36°.

Example 9 (Rectangle and specified triangular shape). Suppose we want to find an isosceles triangle that is isoparametric to a given rectangle. If the isosceles triangle has base c and equal legs of length d, then $d + c/2 = p/2$ and $s = ch/2$, where h is the altitude of the triangle, given by

$$h^2 = d^2 - \left(\frac{c}{2}\right)^2 = \left(d - \frac{c}{2}\right)\left(d + \frac{c}{2}\right).$$

There is no loss in generality in assuming that both the rectangle and isosceles triangle have perimeter 2, which means that $\kappa = 1/s$. Then $d + c/2 = 1$ and $d - c/2 = 1 - c$. Hence $4s = 2ch = 2c\sqrt{1-c}$, and (10) becomes

$$a = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 2c\sqrt{1-c}}, \tag{11}$$

where $0 \le c \le 1$. Figure 8 shows the graph of a, plotted as a function of c. The maximal a occurs for $c = 2/3$ (when the triangle is equilateral) and is given by

$$a_{\max} = \frac{1}{2} - \frac{1}{6}\sqrt{9 - 4\sqrt{3}} = 0.2601\ldots.$$

This upper bound on a also follows directly from (10) by noting that $\kappa \ge 3\sqrt{3}$, the bench mark for all isosceles triangles. It places a constraint on the rectangle parameter a.

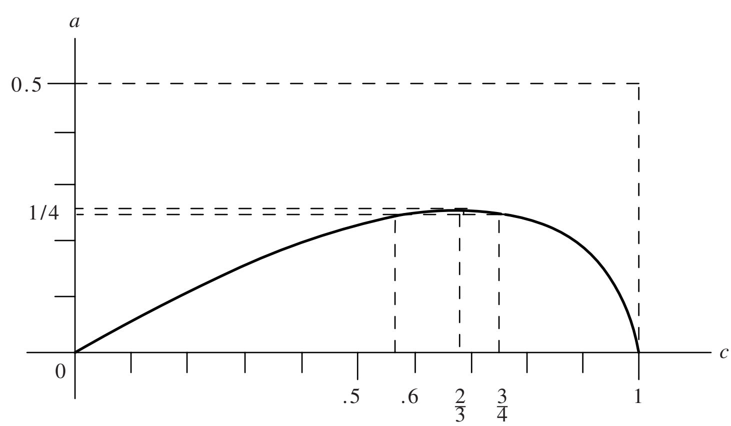


Figure 8. Graph of a as a function of c, where a is given by (11).

For each value of a satisfying $0 < a < a_{\max}$ there are two values of c giving the same a, say $c_1 < c_2$. This means that (with one exception corresponding to $a_{\max}$) there are two different isosceles triangles isoparametric to the rectangle. An example is given in Figure 3b, which shows a rectangle with base 6 and altitude 2, together with two isoparametric isosceles triangles. One of them with base 6, altitude 4, perimeter 16, and area 12, is formed from two Pythagorean $3 : 4 : 5$ triangles. When these are scaled by the factor $1/8$ we get an isosceles triangle with perimeter 2 and area $3/16$, so $c = 3/4$ and $a = 1/4$. This gives the point $(3/4, 1/4)$ on the graph in Figure 8. The line $a = 1/4$ intersects the graph at a second point $(c_0, 1/4)$, where $c_0 = (1 + \sqrt{13})/8 = 0.5757\ldots$. This corresponds to the second isosceles triangle in Figure 3b (scaled by a factor 8).

Similarly, suppose that a rectangle with base b and altitude a is given, and that we seek an isoparametric *right* triangle with base c and altitude h. Again we assume that $a \le b$ and require the perimeter to be 2, so that $a + b = 1$ and $c + h + \sqrt{c^2 + h^2} = 2$. The last equation yields $h = 2(1 - c)/(2 - c)$, and equation (10) becomes

$$a = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4c(1-c)}{2-c}}. \tag{12}$$

The graph of (12), with a plotted as a function of c, resembles that in Figure 8. The maximal a occurs when $c = h = 2 - \sqrt{2}$ (isosceles right triangle) and is given by

$$a_{\max} = \frac{1}{2} - \frac{1}{2}\sqrt{8\sqrt{2} - 11} = 0.21995\ldots.$$

Again, this is a constraint on the rectangle parameter a. For each a with $0 < a < a_{\max}$ there are two values of c giving rise to the same a; they correspond to two right triangles with their legs interchanged.

For small a, the graphs of both (11) and (12) are almost linear with slope $1/2$. There is a geometric reason for this. When a is very small the rectangle's base b is close to 1 and its area is nearly equal to a. And for small c the triangle's altitude h is close to 1 and its area is nearly $c/2$. Equating areas, we have $a \approx c/2$ for small a and c.

Each of the foregoing examples involves shapes described by a parameter that can be adjusted to make two contour ratios equal. This makes the shapes parametrically similar, so they can be scaled to become isoparametric. We turn next to special contours called rings that also depend on a single parameter.

4. RING RATIOS. In this paper we consider the simplest type of ring, the region between two similar simple closed curves with similarity ratio λ, where $0 < \lambda < 1$. An example is shown in Figure 9. We call λ the *size factor* because it measures the size of the inner contour relative to the outer one. If the outer contour has perimeter p and area s, the inner contour has perimeter λp and area $\lambda^2 s$. The inner and outer contours, being similar, have the same contour ratio

$$\kappa = \frac{p^2}{4s}.$$

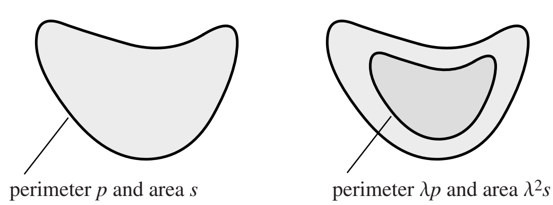


Figure 9. A closed curve with perimeter p and area s used to form a ring with size factor λ.

The ring has its own contour ratio $P^2/(4S)$, where $P = p + \lambda p$ is the total perimeter (the sum of the outer and inner perimeters) and $S = s - \lambda^2 s$ is its area (the difference of the outer and inner areas). The contour ratio of the ring is related to the contour ratio of the inner and outer curves by the equation

$$\frac{P^2}{4S} = \frac{p^2(1+\lambda)^2}{4s(1-\lambda^2)} = \frac{p^2(1+\lambda)}{4s(1-\lambda)} = \kappa\frac{1+\lambda}{1-\lambda}.$$

We call this the *ring ratio* and denote it by ρ to distinguish it from the contour ratio of the boundary curves. Thus we have

$$\rho = \kappa\frac{1+\lambda}{1-\lambda}, \tag{13}$$

where κ is the contour ratio for each closed curve forming the ring.

Similar rings have the same ring ratio and the same size factor. Moreover, if two rings are formed from different boundary curves with the same contour ratio κ, then from (13) we see that for the same size factor λ they also have the same ring ratio.

Equation (13) also shows that

$$\rho > \kappa, \tag{14}$$

and, in fact, the ring ratio is always greater than κ by the factor

$$\frac{1+\lambda}{1-\lambda} = 1 + \frac{2\lambda}{1-\lambda}.$$

From (13) we find, as well, that λ is uniquely determined by ρ and κ:

$$\lambda = \frac{\rho-\kappa}{\rho+\kappa} = 1 - \frac{2\kappa}{\rho+\kappa}. \tag{15}$$

Of course, isoparametric rings have the same ring ratio, but not conversely. In fact, when Theorem 1 is applied to rings we obtain:

Corollary (Isoparametric Ring Theorem). *Two rings can be scaled to become isoparametric if and only if they have the same ring ratio.*

Figure 10 shows various rings arranged as they would appear in Figure 5. Rings of different shapes can have the same ring ratio. Rings joined with sloping lines in Figure 10 have the same size factor λ.

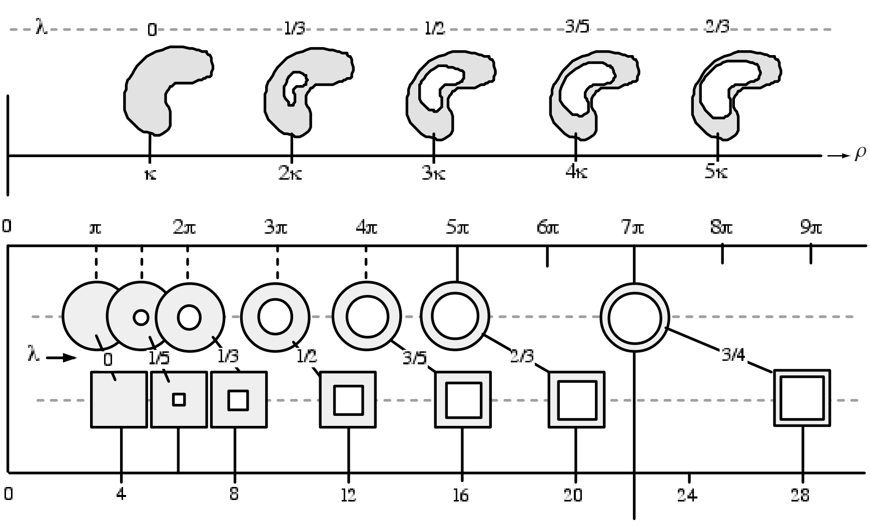


Figure 10. Ring ratios of various rings.

When λ tends to zero, the inner contour shrinks to a point, and (13) shows that the ring ratio ρ becomes the contour ratio κ, which serves as a bench mark with respect to the parameter λ. In particular, the ring ratio of every circular ring is greater than π and can be made arbitrarily close to π by allowing the hole to shrink to a point. In the same way, the ring ratio of every square ring is greater than 4 and can be made arbitrarily close to 4 by allowing the inner square to shrink to a point. Furthermore, (13) implies that $\rho \to \infty$ when $\lambda \to 1$.

5. ISOPARAMETRIC INEQUALITY FOR RINGS. Our next theorem is the ring version of the isoperimetric inequality (1).

Theorem 2. *Among all rings with given size factor λ, the circular ring has the smallest ring ratio. Hence the ring ratio ρ of any ring with size factor λ satisfies*

$$\rho \geq \pi \frac{1+\lambda}{1-\lambda}, \tag{16}$$

with equality only for a circular ring.

Proof. From (13) we infer that among all rings with a given size factor λ the ring ratio is smallest when κ is smallest, and this occurs when $\kappa = \pi$ and the ring is circular. The isoperimetric inequality (1) can be regarded as the limiting case of (16) when $\lambda \to 0$. ■

A general ring ratio ρ as defined by (13) is a function of κ and λ, which we can denote by $\rho_\kappa(\lambda)$. Inequality (16) is a universal inequality, $\rho_\kappa(\lambda) \geq \rho_\pi(\lambda)$, that holds for all rings with given size factor λ. There are also local inequalities of the same type for specified rings. For example, $\kappa = 4$ for a square ring, so the ring ratio of any ring with size factor λ and contour ratio $\kappa \geq 4$ satisfies $\rho_\kappa(\lambda) \geq \rho_4(\lambda)$. And, for all rings formed from contours with given contour ratio κ we also have the inequality $\rho_\kappa(\lambda) > \kappa = \rho_\kappa(0)$.

Inequality (16) tells us when it is possible to have a square ring and circular ring that are isoparametric. We can also see this visually by the display in Figure 10. The ring ratio of every square ring exceeds 4, but there are circular rings with ring ratio arbitrarily close to π, so no square ring has ring ratio in the interval between π and 4. For that interval, the hole in the circular ring is too small to make a significant contribution to the total perimeter and area. This puts a constraint on the circular ring in the form of a lower bound on the size factor. If the size factor exceeds this lower bound, then for any square ring there is a circular ring with a large enough hole to match ring ratios.

To describe this quantitatively, suppose that we are given a circular ring and that we ask for an isoparametric square ring. These rings necessarily have the same ring ratio ρ and, moreover, $\rho > 4$. We know from (15) that the size factor of a circular ring is given by

$$\lambda = 1 - \frac{2\pi}{\rho + \pi} > 1 - \frac{2\pi}{4+\pi} = \frac{4-\pi}{4+\pi}.$$

In other words, the required square ring exists only if the size factor of the circular ring is constrained by the inequality

$$\lambda > \frac{4-\pi}{4+\pi} \approx 0.1202\ldots. \tag{17}$$

This tells us that the hole in the circular ring should have radius slightly more than 12% of the outer radius. Moreover, if λ satisfies (17) then (as will be demonstrated in section 6) there always exists a square ring isoparametric to the circular ring. The example in Figure 3b has $\lambda = 1/2$, which easily satisfies (17). But $\lambda = 1/9$ does not satisfy (17).

When we treat the problem in its general form, a constraint of the form

$$\lambda > \frac{\kappa_2 - \kappa_1}{\kappa_2 + \kappa_1} \tag{18}$$

will appear, generalizing (17). If the rings have equal contour ratios, as do the sectorial ring and rectangular ring in Figure 4a, inequality (18) becomes $\lambda > 0$, which puts no new constraint on λ. This explains the difference between the examples in Figures 4a and 4b.

6. ISOPARAMETRIC RINGS.

We turn next to the following general problem motivated by the example of the square ring and circular ring in section 5:

Isoparametric ring problem. *Given a ring with contour ratio κ_1 and size factor λ_1, under what conditions does there exist an isoparametric ring with given contour ratio κ_2 and some size factor λ_2?*

The key to this problem is equality of the ring ratios:

$$\rho_1 = \kappa_1 \frac{1 + \lambda_1}{1 - \lambda_1}, \qquad \rho_2 = \kappa_2 \frac{1 + \lambda_2}{1 - \lambda_2}.$$

If $\rho_1 \neq \rho_2$, there is no solution. Therefore, we seek conditions ensuring that

$$\kappa_1 \frac{1 + \lambda_1}{1 - \lambda_1} = \kappa_2 \frac{1 + \lambda_2}{1 - \lambda_2}. \tag{19}$$

The problem splits naturally into two cases: $\kappa_1 = \kappa_2$ and $\kappa_1 \neq \kappa_2$.

Case 1: $\kappa_1 = \kappa_2$. In this case, (19) holds if and only if $\lambda_1 = \lambda_2$, in which event the rings can be scaled to become isoparametric. In other words, given any two contours with the same contour ratio (for example, a pentagram and the long narrow rectangle with the same contour ratio mentioned in section 2), we can scale the rectangle to get a similar rectangle isoparametric to the pentagram. This is always possible because of Theorem 1. Using these isoparametric contours as outer contours, we scale each of them by the same size factor $\lambda < 1$ to obtain two isoparametric rings. There are infinitely many solutions because we can use any $\lambda < 1$. Therefore, the case $\kappa_1 = \kappa_2$ presents no difficulties and can be regarded as trivial.

Case 2: $\kappa_1 \neq \kappa_2$. We label the contour ratios so that the smaller one is κ_1. Now we have $\kappa_2 > \kappa_1$, and we want to satisfy (19). If λ_1 is given, where $0 < \lambda_1 < 1$, and we solve (19) for λ_2, then we discover that

$$\lambda_2 = \frac{\kappa_1(1 + \lambda_1) - \kappa_2(1 - \lambda_1)}{\kappa_1(1 + \lambda_1) + \kappa_2(1 - \lambda_1)}. \tag{20}$$

But we also need the inequality $0 < \lambda_2 < 1$. The denominator in (20) is always positive, but the numerator is positive only if $\kappa_1(1 + \lambda_1) > \kappa_2(1 - \lambda_1)$. This puts a constraint on λ_1, namely,

$$\lambda_1 > \frac{\kappa_2 - \kappa_1}{\kappa_2 + \kappa_1}. \tag{21}$$

Therefore, if λ_1 satisfies the constraint (21), then we can always find λ_2 to satisfy (19) and we also have $0 < \lambda_2 < 1$. On the other hand, if λ_2 is given and we solve (19) for λ_1, then we arrive at a companion result to (20),

$$\lambda_1 = \frac{\kappa_2(1 + \lambda_2) - \kappa_1(1 - \lambda_2)}{\kappa_2(1 + \lambda_2) + \kappa_1(1 - \lambda_1)}. \tag{22}$$

Again, we need $0 < \lambda_1 < 1$. The denominator in (22) is positive, and the numerator is positive only if $\kappa_2(1 + \lambda_2) > \kappa_1(1 - \lambda_2)$, which translates to

$$\lambda_2 > \frac{\kappa_1 - \kappa_2}{\kappa_2 + \kappa_1}.$$

This is automatically satisfied because $\kappa_1 - \kappa_2 < 0$, so there is no constraint on λ_2. Therefore, for given λ_2 we can always find λ_1 to satisfy (19), and we get $\rho_1 = \rho_2$. Moreover, from (14) we have $\rho_1 > \kappa_2$, or

$$\kappa_1 \frac{1 + \lambda_1}{1 - \lambda_1} > \kappa_2,$$

which, in turn, is equivalent to (21). This means that when we solve for λ_1 using (22) it automatically satisfies inequality (21). Incidentally, because of (15), the relations (22) and (20) can be expressed more simply in terms of the ring ratio $\rho = \rho_1 = \rho_2$:

$$\lambda_1 = \frac{\rho - \kappa_1}{\rho + \kappa_1}, \qquad \lambda_2 = \frac{\rho - \kappa_2}{\rho + \kappa_2}.$$

The foregoing results are summarized in the following theorem.

Theorem 3. *Two rings with contour ratios κ_1 and κ_2 ($\kappa_2 \geq \kappa_1$) and size factors λ_1 and λ_2 can have the same ring ratio ρ if and only if*

$$\lambda_1 > \frac{\kappa_2 - \kappa_1}{\kappa_2 + \kappa_1}. \tag{23}$$

In this case, the size factor λ_2 is uniquely determined by the equation

$$\lambda_2 = \frac{\rho - \kappa_2}{\rho + \kappa_2}. \tag{24}$$

When $\kappa_2 = \kappa_1$, constraint (23) is automatically satisfied, and (24) gives $\lambda_2 = \lambda_1$. This case is illustrated by the two sectorial rings and the rectangular ring shown in Figure 4a. All three rings have contour ratio 4 and equal size factor λ. Infinitely many pairs of two sectorial rings or of a sectorial ring and a rectangular ring are obtained by allowing λ to vary between 0 and 1. Another such example comes from the pentagram and the special rectangle mentioned in the foregoing proof of Case 1.

As already noted, when the size factor $\lambda \to 0$ the hole in a ring shrinks to a point. The restriction $\lambda > 0$ is imposed on the size factor in order to produce an inner closed curve similar to the outer one. But formula (13) that defines the ring ratio ρ is meaningful if $\lambda = 0$ and gives $\rho = \kappa$ in that case. Consequently, the case $\kappa_1 < \kappa_2$ of Theorem 3 is applicable for the limiting value $\lambda_2 = 0$, in which event the constraint on λ_1 in (23) becomes an equality. And conversely, if $\lambda_1 = (\kappa_2 - \kappa_1)/(\kappa_2 + \kappa_1)$, then the corresponding value of λ_2 is 0. Thus, for example, if $\kappa_2 = 4$ and $\kappa_1 = \pi$, a circular ring with ring ratio 4 and size factor $\lambda_1 = (4 - \pi)/(4 + \pi)$ can be scaled to become isoparametric to a square, a circular sector, or any other contour with contour ratio 4, each of which can be regarded as a limiting case of a ring with $\lambda_2 = 0$. The reader can verify that the following examples, which illustrate Theorem 3, also cover the corresponding limiting cases with $\lambda_2 = 0$ if the constraint inequality in (23) is changed to an equality.

Example 10 (Circular ring and regular polygonal ring). The problem for circular rings and square rings considered in section 5 can be generalized by replacing the square with any regular n-gon. Taking $\kappa_0 = \pi$ and $\kappa_n = n\tan(\pi/n)$, we have $\kappa_n > \kappa_0$, so if a circular ring has size factor λ_0 satisfying

$$\lambda_0 > \frac{n\tan\frac{\pi}{n} - \pi}{n\tan\frac{\pi}{n} + \pi},$$

there is always an isoparametric regular n-gon ring. When $n = 4$ this is inequality (18). As n increases, the lower bound on λ_0 decreases. For example, when $n = 4$ the lower bound is about 0.1202, but when $n = 12$ it is about 0.0116. This indicates that the radius of the hole in the circular ring needs to be at least 12% of the outer radius to have an isoparametric square ring, but 1.2% suffices for a dodecagonal ring. There is more latitude in finding isoparametric dodecagonal rings because a dodecagon is more "circular" than a square.

Example 11 (Pythagorean 3 : 4 : 5 triangular ring and square ring). Consider a triangular ring bounded by two similar Pythagorean 3 : 4 : 5 triangles. An example with outer triangle of edges 9, 12, and 15 and size factor 1/3 is depicted in Figure 11. All such rings have contour ratio $\kappa = 6$ and ring ratio $\rho > 6$. An isoparametric square ring with inner side-length a and outer side-length b has contour ratio 4 and the same ring ratio ρ. Since $4 < 6$, we label the square ring as ring 1 and the triangular ring as ring 2. Constraint (23) states that isoparametric square rings exist if they have size factor $\lambda_2 > (6 - 4)/(6 + 4) = 1/5$. The square ring in Figure 11 has $\lambda_2 = 1/2$, which suffices. The triangular ring has perimeter 48, so equality of total perimeters dictates that $48 = 12a$, and we find that $a = 4$, $b = 8$. On the other hand, no square ring with size factor smaller than 1/5 is isoparametric to any 3 : 4 : 5 triangular ring.

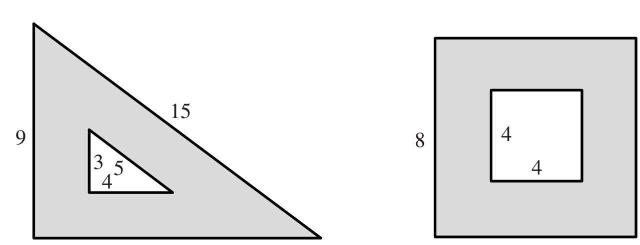


Figure 11. A Pythagorean 3 : 4 : 5 triangular ring and a square ring that are isoparametric.

Example 12 (Two regular polygonal rings). This example compares a ring formed by two regular n-gons with one formed by two regular m-gons, where $n > m$. The contour ratio for the n-gon is $\kappa_n = n\tan(\pi/n)$, while that for the m-gon is $\kappa_m = m\tan(\pi/m)$. Because κ_n is a decreasing function of n, we have $\kappa_n < \kappa_m$, and constraint (23) becomes

$$\lambda_n > \frac{m\tan(\pi/m) - n\tan(\pi/n)}{m\tan(\pi/m) + n\tan(\pi/n)}, \tag{25}$$

where λ_n and λ_m replace λ_1 and λ_2 in Theorem 3. From the polynomial approximation $\tan x \sim x + x^3/3$, valid for small x, we see that for large m and n the quotient on the right of (25) has the asymptotic value

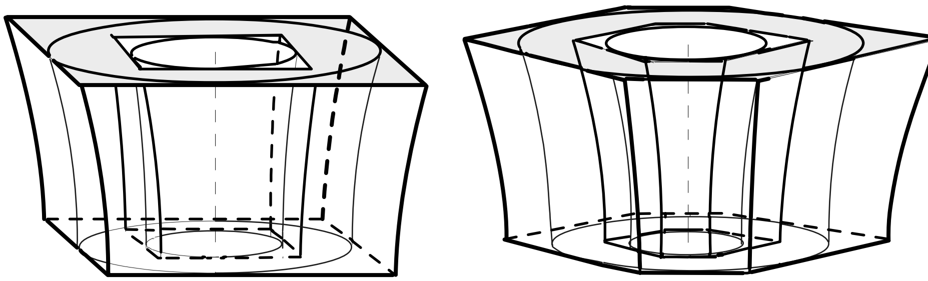
$$\frac{\pi^2}{6}\left(\frac{1}{m^2} - \frac{1}{n^2}\right).$$

7. ISOPARAMETRIC RINGS WITH EQUAL INNER PERIMETERS AND EQUAL OUTER PERIMETERS. Two rings that are isoparametric have the same area and the same total perimeter. We can also construct such rings in which both inner perimeters are equal and both outer perimeters are equal. For example, take two different polygons that circumscribe the same circle and have the same perimeter. Then they are isoparametric and have the same contour ratio. Now scale each polygon by the same size factor λ to produce two polygonal rings that are isoparametric. These rings have the additional property that both the inner perimeters are equal and the outer perimeters are equal. In fact, we have an entire family of such isoparametric polygonal rings, one ring for each λ. This property will be used in section 8 to generate a remarkable family of incongruent solids satisfying the six properties (a) through (f) listed in section 1.

More generally, take any two incongruent isoparametric contours bounded by simple closed curves. Scale each of them by the same size factor λ to produce two incongruent isoparametric rings. Then these rings also have the additional property that both the inner perimeters are equal and the outer perimeters are equal. Figure 4a shows such an example. We leave it to the reader to verify that a necessary and sufficient condition for two isoparametric rings to have both the inner perimeters equal and the outer perimeters equal is that both the two outer contours be isoparametric and the two inner contours be isoparametric.

8. INCONGRUENT SOLIDS HAVING PROPERTIES (a) THROUGH (f). This section describes several families of incongruent solids having the six properties (a) through (f) listed in the introduction. Each family is generated by isoparametric circumscribing polygons of the type discussed in section 7.

Start with a smooth solid of revolution whose cross sections by horizontal planes perpendicular to the rotation axis are circular rings. Take any polygonal ring of the type discussed in section 7 that circumscribes the base, and use a similar polygonal ring to circumscribe each parallel circular cross section above the base. The union of all these polygonal rings sweeps out a solid, an example of which is shown in Figure 12a. Repeat the process, starting with a noncongruent isoparametric polygonal ring on the base to produce a noncongruent solid like the example in Figure 12b. In each horizontal cross section the two polygonal rings are isoparametric and circumscribe the same circle. Moreover, the two inner polygons have equal perimeters, as do the two outer polygons. Equality of cross-sectional areas implies equality of volumes of the



(a) (b)

Figure 12. Incongruent solids sharing the six properties (a) through (f) of section 1.

two solids, and, because the perimeters are equal, it is not difficult to prove that both the inner and outer lateral surface areas are equal. In each solid, the inner lateral surface is similar to the outer lateral surface by some size factor λ. These families share a seventh property, which states that for each solid the intermediate lateral surfaces corresponding to the same choice of size factor between 0 and λ have equal areas.

We can choose the polygonal rings in infinitely many ways, so for each given solid of revolution we have infinitely many pairs of incongruent solids satisfying properties (a) through (f). And we can generate more such families by starting with different solids of revolution. In particular, when the solid of revolution is a hemisphere, these solids are related to Archimedean shells, which are discussed in [**1**]. In this case, two different Archimedean shells with isoparametric polygonal bases circumscribing congruent equators of two hemispheres provide examples of noncongruent solids sharing properties (a) through (f).

9. FURTHER RESULTS. The investigations in this paper suggest a host of complex and interesting problems that we will pursue in a sequel. First, we can find isoparametric rings such that the "hole" in each ring has a shape different from that of the outer contour. Although the results of this paper deal with holes similar to the outer contour, they can also be extended to treat isoparametric holes that are dissimilar.

Second, most of the results can be extended to higher-dimensional space, which allows many ways to extend isoparametric problems. For example, in 3-space we can compare volumes and surface areas, or we can compare surface areas and linear sizes such as edge-lengths and perimeters, or volumes and linear sizes. In higher-dimensional space we can compare n-dimensional and m-dimensional volumes and sizes.

Third, instead of requiring that the perimeters and areas of two plane regions be equal, we could ask that they have prescribed ratios. This more general situation can be treated using the theorems of this paper. And there are similar extensions in higher-dimensional space.

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