The method of sweeping tangents

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I. INTRODUCTION: BASIC THEORY AND EXAMPLES

Sweeping tangents

What is the area of the shaded region between the tyre tracks of a moving bicycle such as that depicted in Figure 1? If the tracks are specified, and equations for them are known, the area can be calculated using integral calculus. Surprisingly, the area can be obtained more easily without calculus, regardless of the bike's path, using a dynamic visual approach called *the method of sweeping tangents* that does not require equations for the curves.

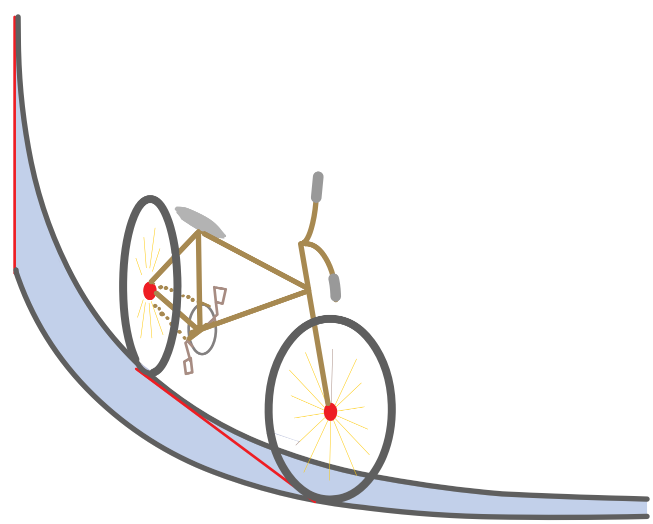

Figure 1: What is the area of the region between the tyre tracks of a bicycle?

The method is illustrated in Figure 2. The segment joining the bottom of the rear wheel with that of the front wheel has constant length, denoted here by k, and is always tangent to the path of the rear wheel. As the bicycle moves, this tangent

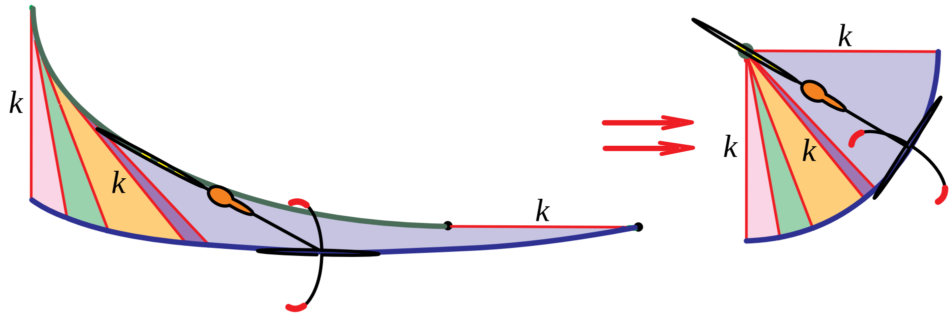

Figure 2: The area of the region between the tracks is that of a quarter of a circular disk.

segment sweeps out the shaded region between the tyre tracks, as suggested by the left part of Figure 2. The right part of Figure 2 shows each tangent segment translated (parallel to itself) to bring the points of tangency together at a single point. Because the tangents have constant length k, the translated segments sweep out a circular sector of radius k. Therefore, the area of the shaded region between the tracks is equal to the area of this circular sector, which depends only on the length k and the change in angle from the bike's initial position to its final position. To convince yourself that the areas are equal, consider corresponding tiny "triangles" of equal area translated from left to right, as suggested in Figure 2.

Mamikon's sweeping-tangent theorem

The method of sweeping tangents extends this idea to more general curves and tangent segments of variable length, as shown in Figure 3. Begin with a smooth curve τ, called the *tangency curve*, together with a moving tangent line. The point of tangency moves along τ in a given direction, called the positive direction, as indicated by the arrowhead in Figure 3. At each point of τ the tangent line defines two rays, one in the positive direction of motion, the other in the opposite direction. It may be helpful to imagine an automobile driving along τ with its headlight beam indicating the direction of a tangent ray. If the automobile moves forward in the positive direction its headlight beam indicates the direction of motion. If it drives backward, the headlight beam points in the opposite direction.

Assume a tangent vector moves continuously, always pointing in the positive direction during the motion, or else always pointing in the opposite direction. The moving tangent vector sweeps out a region called the *tangent sweep*. The free end of

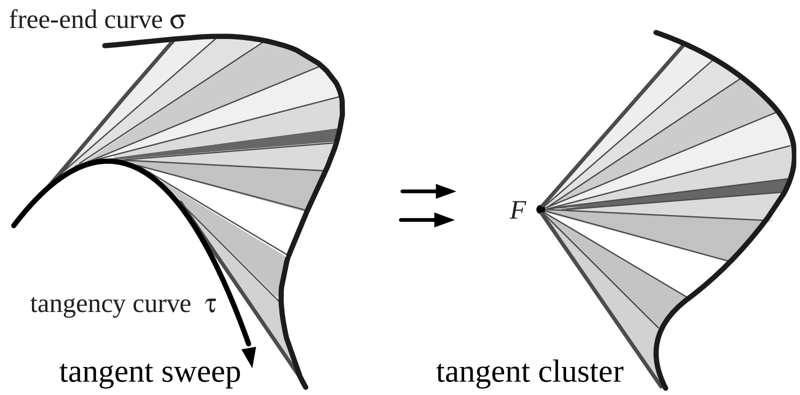

Figure 3: The sweep of a tangent of variable length and its cluster have equal areas.

the tangent vector traces a curve σ called the *free-end curve.* There are two possible free-end curves, σ_+ generated when the tangent vectors point in the positive direction, and σ_- generated when the tangent vectors point in the backward direction. (For an animated version of two free-end curves see the website in [1] and click on DualRing.)

When each tangent vector is translated, parallel to itself, to bring the points of tangency together at a single point F, the set of translated segments is called a *tangent cluster.* Figure 3 shows a tangent sweep (left) and its corresponding tangent cluster (right). The method of sweeping tangents is based on:

Mamikon's sweeping-tangent theorem. *The area of a tangent sweep is equal to the area of its tangent cluster, regardless of the tangency curve.*

Note that the tangent cluster of σ_- is a reflection through F of σ_+, so they have equal areas. Mamikon's theorem and its extension to 3-space is proved in [7] (in Russian); a different proof, given in [2], is outlined briefly at the end of this paper.

For tangents of constant length, the method of sweeping tangents reveals the striking property that the area of the tangent cluster is always equal to the area of a circular sector, regardless of the tangency curve τ. But the most striking applications are those with tangent segments of variable length. They reveal the true power of the method, which yields areas of regions below the graphs of exponential functions, power functions, cycloids, and many other classical curves that are described by their geometric properties rather than by equations. They are discussed in [1]-[4] and [7].

First we briefly summarize the method of sweeping tangents for area, and then we turn to the principal goal of this paper, which is to extend the method to the more difficult problem of finding arclengths of these classical curves.

Area of the tangent cluster

Figure 4 shows a close-up view of the tangent sweep introduced in Figure 3. The tangent vector from τ to σ points in the positive direction. We denote the length of this vector by $t = t(\alpha)$, where α is the angle between the moving tangent and some initial direction, for example, the tangent direction at a conveniently chosen point denoted by O in Figure 4. The function $t(\alpha)$ provides a polar description of the tangent cluster, with $t(\alpha)$ representing the radial distance from the common point F to the boundary curve of the cluster in Figure 3. The proof of Mamikon's theorem reveals that the area of the tangent sweep is equal to that of the tangent cluster which, in turn, can be expressed as an integral in polar coordinates, where α varies from an initial value α_1 to a larger value α_2:

$$\text{area of tangent sweep} = \text{area of tangent cluster} = \frac{1}{2}\int_{\alpha_1}^{\alpha_2} t^2(\alpha)d\alpha. \tag{1}$$

In applying the method of sweeping tangents, we try to construct the tangent lengths in such a way that the area of the cluster is easy to calculate. In the bicycle problem in Figure 2, the tangent segments have constant length k, and the integral (1) gives the same result we found geometrically:

$$\text{area} = \frac{1}{2}k^2\theta, \tag{2}$$

which is the area of a circular sector of radius k subtending angle $\theta = \alpha_2 - \alpha_1$.

Arclength of tangency curve: tangent in the forward direction

Next we show that a knowledge of $t(\alpha)$ enables us to relate the arclength l of τ in terms of angle α by formulas analogous to (1), as described in (4) and (5) below. Each of these formulas provides an *intrinsic equation* for τ, one that does not rely on any external coordinate system, such as rectangular or polar coordinates. The intrinsic equation of a curve is also known as its *natural equation.*

Figure 4 shows the tangent segment of length $t = t(\alpha)$ at position A and at a nearby position B as α changes by $\Delta\alpha$. The point of tangency slides from A to B along τ through distance Δl, and the free end of the tangent vector traces a small arc Δs on σ. We treat arcs Δs and Δl as linear approximations to the curves, as depicted in Figure 4 for small $\Delta\alpha$. Thus, arc Δs is the hypotenuse of a right triangle, one leg of which is $t(\alpha)\Delta\alpha$, due to rotation of the tangent vector through angle $\Delta\alpha$ without changing its length. The other leg is made up of two parts, $\Delta l + \Delta t$, where Δl is caused by sliding the rotated tangent along τ without changing its length, and Δt is caused

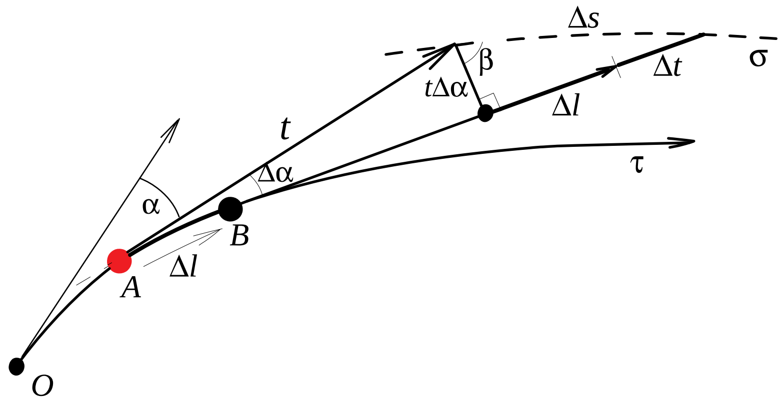

Figure 4: Arclength relations determined by two nearby tangents.

by the variability of its length. In what follows, we always assume that arclength l increases as the point of tangency moves along τ, so Δl is positive. However t may increase or decrease during the motion, so Δt can be positive or negative. In Figure 4 the tangent vector points in the forward direction, and Δt is shown as positive.

For a given α, let β denote the complement of the angle between the two tangents to τ and σ. Thus β is a function of α. The triangle ratio for tan β in Figure 4 yields the following approximate relation:

$$\Delta l + \Delta t \approx t(\alpha)\Delta\alpha \,\tan\beta.$$

Divide by $\Delta\alpha$ and let $\Delta\alpha \to 0$ to obtain the following equation relating the derivatives of l and t:

$$\frac{dl}{d\alpha} + \frac{dt}{d\alpha} = t(\alpha)\tan\beta. \tag{3}$$

Integrating (3) from $\alpha = 0$ to $\alpha = \theta$ we find an intrinsic equation for l in terms of t:

$$l(\theta) - l(0) + t(\theta) - t(0) = \int_0^\theta t(\alpha)\tan\beta\, d\alpha. \tag{4}$$

In most applications we choose O so that $l(0) = 0$.

Arclength of tangency curve: tangent in the backward direction

In some applications it is convenient to reverse the direction of the tangent vector so it points opposite to that of increasing l, as shown in Figure 5. Then the triangle ratio for tan β becomes $|\Delta l - \Delta t| \approx t(\alpha)\Delta\alpha \,\tan\beta$, where the absolute value allows

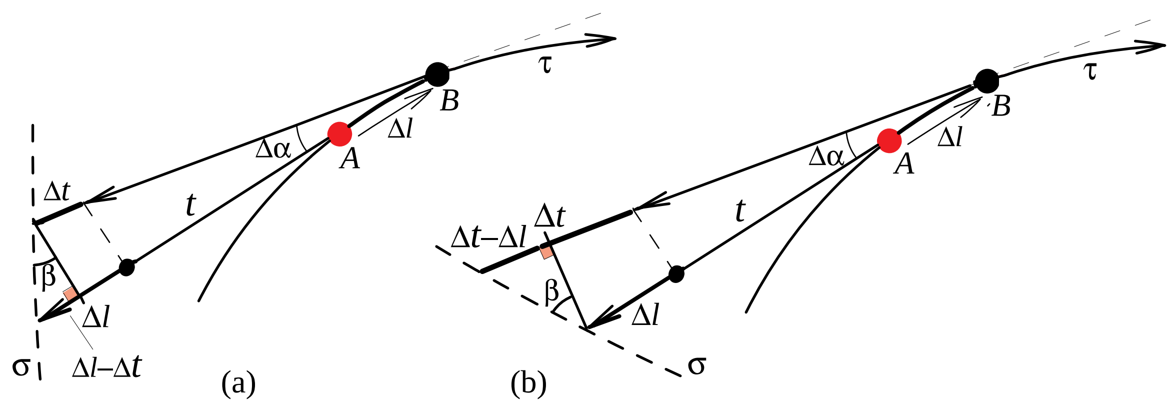

Figure 5: Arclength relations when the tangent vector has backward direction.

for the two possibilities: $\Delta l > \Delta t$ (Figure 5a), and $\Delta t > \Delta l$ (Figure 5b). This leads to a corresponding change in (3):

$$|\frac{dl}{d\alpha} - \frac{dt}{d\alpha}| = t(\alpha)\tan\beta,$$

and, instead of (4), we now have the backward integrated version

$$|l(\theta) - l(0) - t(\theta) + t(0)| = \int_0^\theta t(\alpha)\tan\beta\, d\alpha. \tag{5}$$

Guided by the automobile analogy, we call (4) the *forward* relation and (5) the *backward* relation. They are identical when $t(\alpha)$ is constant.

Arclength of free-end curve

Let Δs denote the change in arclength s of free-end curve σ (the hypotenuse of the small triangle in Figure 4), measured so that $s = 0$ when $\alpha = 0$. Then we have the following approximate relation, which also holds for Figure 5:

$$\Delta s \approx \frac{t(\alpha)\Delta\alpha}{\cos\beta},$$

which gives us

$$\frac{ds}{d\alpha} = \frac{t(\alpha)}{\cos\beta}. \tag{6}$$

Integrating (6), we obtain an intrinsic equation for s in terms of t:

$$s(\theta) = \int_0^\theta \frac{t(\alpha)}{\cos\beta}\, d\alpha. \tag{7}$$

Relating arclengths of free-end curve and tangency curve

A direct connection between the arclengths l of τ and s of σ can be found by eliminating β in the basic derivative relations (3) and (6). Square each of these and use the identity $\tan^2\beta = \sec^2\beta - 1$ to obtain the relation

$$(\frac{dl}{d\alpha} + \frac{dt}{d\alpha})^2 = (\frac{ds}{d\alpha})^2 - t^2(\alpha), \tag{8}$$

which involves intrinsic equations of τ and σ. It can also be derived directly by applying the Pythagorean theorem to the triangle with hypotenuse Δs in Figure 4. Relation (8), in turn, gives an explicit formula expressing s in terms of l:

$$s(\theta) = \int_0^\theta \sqrt{t^2(\alpha) + (\frac{dl}{d\alpha} + \frac{dt}{d\alpha})^2}\, d\alpha. \tag{9}$$

It is easy to show that the classical arclength integral in polar coordinates is a limiting case of (9). Take τ to be a small circular arc that shrinks to a point. Then $t(\alpha)$ becomes the radial distance r from this point to σ, $dl/d\alpha \to 0$, and (9) becomes

$$s(\theta) = \int_0^\theta \sqrt{r^2 + (\frac{dr}{d\alpha})^2}\, d\alpha.$$

Also, (9) can be transformed to resemble the classical integral in rectangular coordinates,

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx,$$

for the arclength of a curve $y = f(x)$ between two points $(a, f(a))$ and $(b, f(b))$. This follows from (9) by the substitution $dx = t(\alpha)d\alpha$ and $dy = dl + dt$. The orthogonality of dx and dy is revealed by the small right triangle in Figure 4.

One can also use (8) to formulate a counterpart to (9) expressing l in terms of s.

Examples: Tangents of constant length

All such examples are special cases of the bicycle problem, with constant length $t(\alpha) = k$. For any choice of τ and σ, the area of the tangent sweep is given by (2), both arclength formulas (4) and (5) become

$$l(\theta) - l(0) = k \int_0^\theta \tan\beta\, d\alpha, \tag{10}$$

and (7) takes the form

$$s(\theta) = k \int_0^\theta \frac{1}{\cos\beta}\, d\alpha.$$

Now we consider several examples by specializing τ and σ.

Oval rings. A tangent segment of constant length k moving once around a smooth simple closed plane curve τ sweeps out a region called an *oval ring*. The area of the oval ring is equal to πk^2 because its tangent cluster is a circular disk of radius k. If the tangent moves part way through an angle $\theta \leq 2\pi$ the tangent cluster is a circular sector of area $\frac{1}{2}k^2\theta$, as depicted in Figure 6.

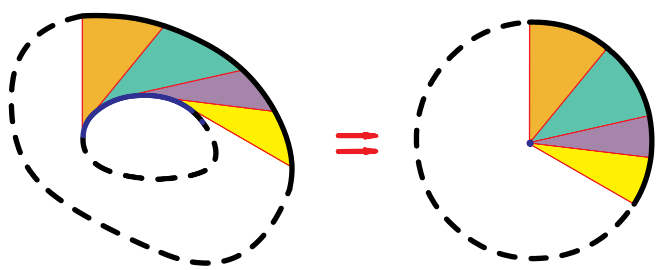

Figure 6: A swept portion of an oval ring has the same area as a circular sector.

Circular rings. Consider a subcase when τ is a circle of radius r and free-end curve σ is a concentric circle of radius R. The area of the circular annulus is the area of the cluster πk^2, and is also equal to the difference $\pi R^2 - \pi r^2$. Equating these areas we find the Pythagorean relation, $k^2 = R^2 - r^2$. Thus, the Pythagorean theorem is a consequence of Mamikon's sweeping-tangent theorem.

For arclengths of concentric circles we use (10) and (7). From Figure 7a we find $\tan\beta = r/k$ and $\cos\beta = k/R$. Taking $l(0) = 0$, we obtain the familiar formulas for the length of a circular arc subtended by an angle θ:

$$l(\theta) = r\theta, \quad s(\theta) = R\theta.$$

To demonstrate the flexibility in the choice of free-end curve we now choose σ to be a line through the center of the circle τ of radius r as in Figure 7b. In this example, $t(\alpha)$ is not constant, but $t(\alpha) = r\tan\alpha$ and $l(\alpha) = r\alpha$, so

$$t(\theta) - l(\theta) = r(\tan\theta - \theta).$$

On the other hand, from the backward relation (5) we also find (because now $\beta = \alpha$)

$$t(\theta) - l(\theta) = r\int_0^\theta \tan^2\alpha\, d\alpha.$$

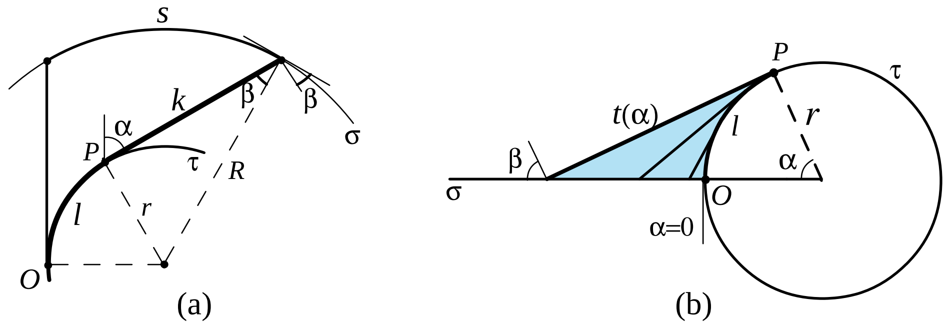

Figure 7: (a) Circular arclength. (b) Alternative choice of σ as a line through the center.

Therefore, as a fringe benefit of this analysis we obtain the known integration formula

$$\int_0^\theta \tan^2 \alpha \, d\alpha = \tan\theta - \theta. \tag{11}$$

We can also derive (11) using the area of the tangent sweep in Figure 7b, which (when $\alpha = \theta$) is that of a right triangle of edges r and $t(\theta) = r\tan\theta$, minus the area of the circular sector subtending angle θ. But this area is also that of the tangent cluster which, by (1), equals $\frac{1}{2}r^2 \int_0^\theta \tan^2 \alpha \, d\alpha$, and again we obtain (11).

Tractrix. Now take the tangency curve τ to be a *tractrix*, the trajectory of a toy on a taut string being pulled by a child walking along a linear path, which we take as σ, as shown in Figure 8 (left). A standard calculus problem is to calculate the area of the entire region between the tractrix and the line. Use of calculus requires finding a Cartesian equation for the tractrix, which in itself is a challenging problem that usually involves solving a differential equation. Once the Cartesian equation

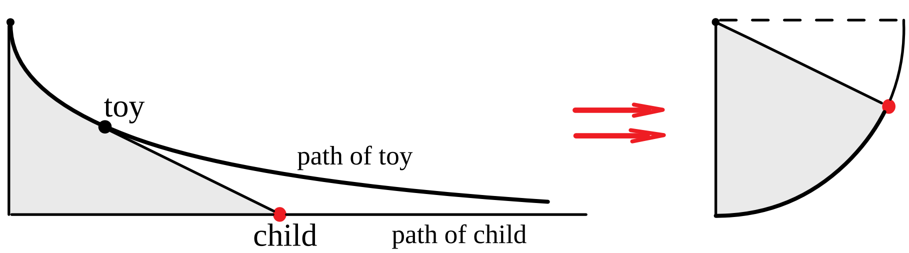

Figure 8: Tangent sweep for a tractrix. The tangent cluster is a circular sector.

is known, integration (which is tedious) shows that the area of the entire region is simply $\pi k^2/4$, where k is the length of the string. But the same result can be found at once without calculus because the tractrix is a particular case of the "bicyclix," whose tangent cluster is a circular sector of radius k, shown shaded in Figure 8 (right). The area of the entire region is that of a quarter of a circular disk, as in Figure 2.

For arclength, refer to Figure 9. The tangency curve τ is a tractrix, σ is the positive x axis, the tangent segment has constant length k, $\beta = \alpha$, and $l(0) = 0$, hence (4) becomes

$$l(\theta) = k \int_0^\theta \tan\alpha \, d\alpha = -k \log(\cos\theta). \tag{12}$$

This is an intrinsic equation for the tractrix, expressing its arclength in terms of θ.

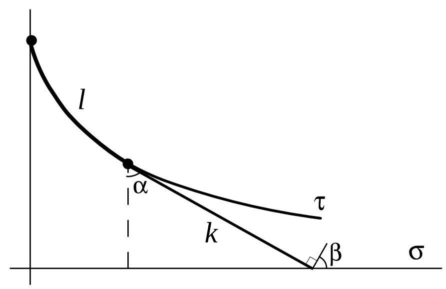

Figure 9: Arclength of tractrix.

If the point of tangency has Cartesian coordinates (x, y) when $\alpha = \theta$, then $\cos\theta = y/k$, and (12) gives the arclength $L = L(y)$ of the tractrix from $(0,k)$ to (x, y) as

$$L = -k \log \frac{y}{k}.$$

From this we find $y = ke^{-L/k}$, which means the ordinate y decreases exponentially with L. In particular, if an automobile drives along the tractrix with constant speed, its distance from the x axis decreases exponentially with time.

Note: The arclength formula (7) for the free-end curve yields an unexpected fringe benefit for the tractrix. Using $\beta = \alpha$ and $t(\alpha) = k$ in (7), we find

$$s(\theta) = \int_0^\theta \frac{k}{\cos\alpha} \, d\alpha = k \log \frac{1 + \sin\theta}{\cos\theta}.$$

Let (x, y) be the point of tangency on the tractrix when $\alpha = \theta$. Because σ is the x axis, we find

$$x = s(\theta) - k\sin\theta, \quad y = k\cos\theta.$$

Using $\cos\theta = y/k$, we find the classical Cartesian representation of the tractrix as a direct consequence of intrinsic equation (7):

$$x = k \log \frac{k + \sqrt{k^2 - y^2}}{y} - \sqrt{k^2 - y^2}.$$

In more general examples, where $t(\alpha)$ is not constant, whenever σ is a straight line we can take it to be the x axis and measure α from a vertical line. From a right triangle like that in Figure 9, the intrinsic equation for s as a function of α yields

$$x = s(\alpha) - t(\alpha)\sin\alpha, \quad y = t(\alpha)\cos\alpha,$$

which are parametric equations of the tangency curve τ in rectangular coordinates.

II. APPLICATIONS: TANGENT SEGMENTS OF VARIABLE LENGTH

Now we consider examples with tangent segments of variable length, and discuss both areas of tangent sweeps and arclengths of classical curves described by geometric properties rather than by equations. Our results for arclength provide intrinsic equations for these curves.

Exponential. In Figure 10 the tangency curve τ is the graph of an exponential $y = e^{x/b}$, where b is a positive constant, and the free-end curve σ is the x axis. It is known (see

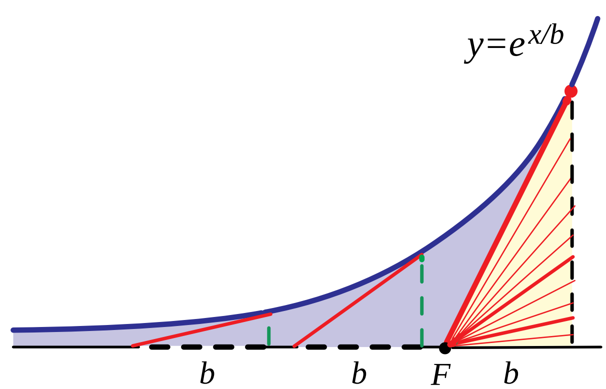

Figure 10: Region under an exponential curve swept by tangents.

[4]) that exponential curves are the only curves with constant subtangents. In fact, the exponential curve in Figure 10 has constant subtangents of length b.

We use this geometric property to find the area of the region under an exponential curve without integral calculus. Part of this region, shown shaded in Figure 10, is swept by the tangent segments cut off by the x axis as the point of tangency moves left from (x, y) to $(-\infty, 0)$. Instead of using (1) to find the area as an integral, we refer to Figure 10, which shows each tangent segment translated so that the endpoint on σ (the x axis) is brought to a common point F, namely, the lower vertex of a right triangle as shown. Because the subtangent is constant, the resulting tangent cluster forms a right triangle of base b and altitude y, whose area is also the area of the tangent sweep. Consequently, the area of the region between the exponential curve and the interval $(-\infty, x]$ is twice the area of the right triangle, which is the area of a rectangle of base b and altitude y, or by.

In the language of calculus, we have shown that

$$\int_{-\infty}^{x} e^{u/b} du = be^{x/b},$$

but we obtained this without the formal machinery of integral calculus. The only property of the exponential that we used was the constancy of the subtangent.

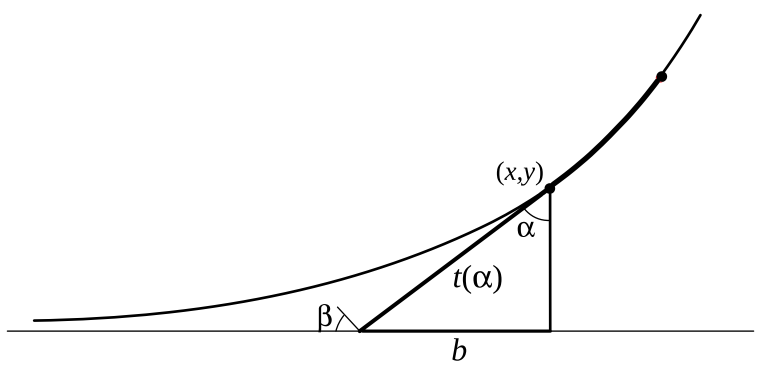

Figure 11: Arclength of exponential.

For arclength, refer to Figure 11 which shows that $t(\alpha) = b/\sin\alpha$. Again we have $\beta = \alpha$, hence $t(\alpha)\tan\beta = b/\cos\alpha$. Integrate (3) from θ_0 to θ and denote the arclength by $l(\theta)$ to find

$$l(\theta) = b(\frac{1}{\sin\theta_0} - \frac{1}{\sin\theta} + \log\frac{1+\sin\theta}{\cos\theta} - \log\frac{1+\sin\theta_0}{\cos\theta_0}). \tag{13}$$

This intrinsic equation for the exponential is valid for $0 < \theta_0 < \theta < \pi/2$.

At a general point of tangency (x, y) with angle α we have

$$t(\alpha) = \sqrt{b^2 + y^2}, \ \sin\alpha = \frac{b}{\sqrt{b^2 + y^2}}, \ \cos\alpha = \frac{y}{\sqrt{b^2 + y^2}},$$

and the intrinsic equation (13) gives the classical formula for the arc length L of the exponential $y = e^{x/b}$ between (x_2, y_2) and (x_1, y_1), where $y_1 > y_2$:

$$L = \sqrt{b^2 + y_1^2} - \sqrt{b^2 + y_2^2} + b \log \frac{\sqrt{b^2 + y_2^2} + b}{y_2(\sqrt{b^2 + y_1^2} + b)}.$$

Parabola. Figure 12a shows a portion of a parabola $y = x^2$ above the interval $[0, x]$. We take the parabola as tangency curve τ and the x axis as free-end curve σ. The tangent segment from τ at (x, x^2) to σ has subtangent $x/2$. From this property, we have shown in two different ways (using Figures 12a and 12b) that the area of the parabolic segment between τ and σ is $x^3/3$. (See [1]-[3], where the general power function $y = x^n$ is also treated without integration.)

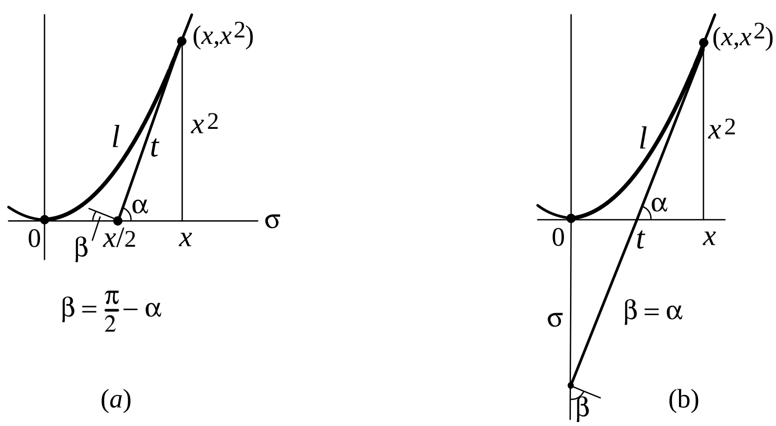

Figure 12: Two methods for calculating the arclength of a parabola.

Now we calculate the arclength traced by a point moving along the parabola from the origin to (x, x^2). To demonstrate the flexibility in the choice of free-end curve, Figures 12a and 12b illustrate two different ways for calculating the arclength. In both cases, α denotes the angle between the tangent at (x, x^2) and the x axis, but in Figure 12b the end-curve σ is chosen to be the negative y axis.

In the backward formula (5) for arclength, $t(\alpha)\tan\beta$ appears in the integrand. To express $t(\alpha)\tan\beta$ in terms of α, note that in Figure 12a we have $\beta = \pi/2 - \alpha$, and hence $\tan\beta = \cot\alpha$. Also, $\tan\alpha = x^2/(x/2) = 4(x/2) = 4t(\alpha)\cos\alpha$, so that

$$t(\alpha) = \frac{\tan\alpha}{4\cos\alpha}, \text{ and } t(\alpha)\tan\beta = \frac{1}{4\cos\alpha}.$$

Now we use (5) with $l(0) = t(0) = 0$ and we find that the arclength of the parabola is given by

$$l(\theta) = t(\theta) + \frac{1}{4}\int_0^\theta \frac{d\alpha}{\cos\alpha} = \frac{1}{4}\Big(\frac{\tan\theta}{\cos\theta} + \log\frac{1+\sin\theta}{\cos\theta}\Big). \tag{14}$$

Figure 12b leads to the same formula for arclength. Now $\beta = \alpha$ and $t(\alpha)$ has twice the value in the foregoing calculation. In this case $t(\alpha) > l(\alpha)$ and a sign change is required in (5). We omit the details. The arclength of a parabola was first found by Isaac Barrow, who used a different method and expressed the result in an equivalent form involving $\theta/2$. Again, (14) represents an intrinsic equation for the parabola.

In terms of x, the length $L(x)$ of the parabolic arc from the origin to (x, x^2) is

$$L(x) = \Big|\frac{1}{2}x\sqrt{4x^2+1} + \frac{1}{4}\log(2x + \sqrt{4x^2+1})\Big|.$$

Cycloid. A cycloid is the path traced by a fixed point on the boundary of a circular disk that rolls along a horizontal line (Figure 13). For example, a light fastened to the rim of a bicycle wheel traces a cycloid as the wheel rolls along a horizontal line. When the wheel turns half way it traces out cycloidal arc OM. (Another half turn produces its mirror image.) In [1] we used the method of sweeping tangents to derive the classical result that the area of the region under the cycloidal arc OM is three halves times that of the rolling disk. This is based on the fact that the tangent sweep $OPTA$ in Figure 13a has the circular wedge TPC as tangent cluster. When $P = M$ the tangent cluster is a semicircular disk of diameter D whose area is $\pi D^2/8$. The circumscribing rectangle in Figure 13a has area $\pi D^2/2$ which implies that the area of the region under cycloidal arc OM is three halves times that of the rolling disk.

For arclength we refer to Figure 13b. Tangency curve τ is a cycloid, and the free-end curve is chosen as a horizontal line tangent to the highest point M of the cycloid, at distance D above the base. Tangent length $t(\alpha)$ is one leg of a right triangle inscribed in a semicircle of diameter D, so $t(\alpha) = D\cos\alpha$. Now $\beta = \alpha$ and $t(\alpha)\tan\beta = D\sin\alpha$. Formula (4) (with θ replaced by α) leads to

$$l(\alpha) = 2D(1-\cos\alpha) = 2D - 2t(\alpha).$$

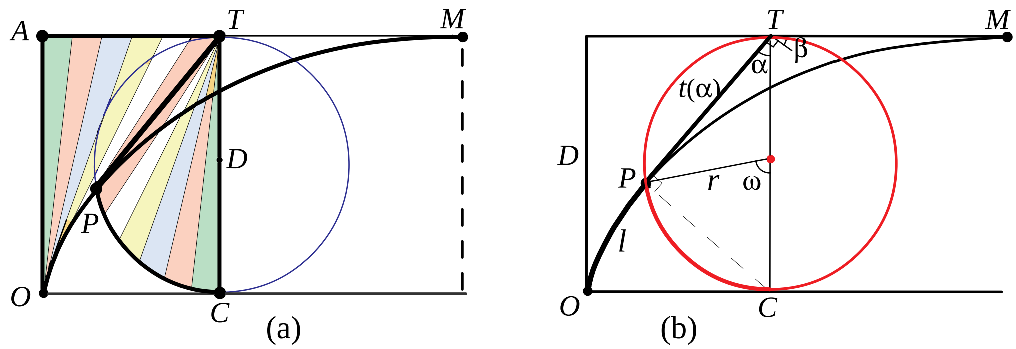

Figure 13: (a) Area of cycloidal tangent sweep $OPTA$ is equal to that of circular wedge TCP. (b) Length of cycloidal arc PM is twice that of tangent segment PT.

In particular, when $\alpha = \pi/2$, arclength OM is $2D$, a result discovered by Christopher Wren. The general formula for $l(\alpha)$ implies that arclength PM, which is $2D - l(\alpha)$, is twice the length of tangent segment PT.

In Figure 13b, $D = 2r$, where r is the radius of the rolling disk that traces the cycloid. Denote by $L(\omega)$ the length of cycloidal arc OP in terms of the angle of turn ω of the rolling disk. Then the formula for arclength OP becomes

$$L(\omega) = 4r(1 - \cos\frac{\omega}{2}). \tag{15}$$

Epicycloid and hypocycloid. Figure 14a shows an epicycloid, a curve traced by a point P on the boundary of a disk of radius r that rolls along the outer circumference of a fixed circle of radius R. When the rolling disk makes one complete turn it generates an arch outside the fixed circle. The method of sweeping tangents, together with Figure 14, can be used to show that the area of this arch is κ times that of the rolling disk, where $\kappa = 1 + 2r/R$. The argument is based on the fact that the tangent sweep $OPTA$ in Figure 14a has the shaded region in Figure 14b as tangent cluster. This is part of a rosette whose area, in turn, is κ times that of the shaded circular wedge in Figure 14c. Both epicycloidal and hypocycloidal areas are treated in [6] by a different elementary method.

Here we investigate arclength. In this case, $\alpha = \kappa\beta$, where again $\kappa = 1 + 2r/R$. In Figure 14a the tangency curve τ is the epicycloid and the free-end curve σ is the circular arc of radius $R + 2r$. We see that $PT = t(\alpha) = 2r\cos\beta$ and the integrand in (4), when expressed in terms of β, becomes $2r\kappa\sin\beta\, d\beta$. Now $\beta = \omega/2$, where ω is the angle of turn of the rolling disk of radius r. In terms of ω we find the following

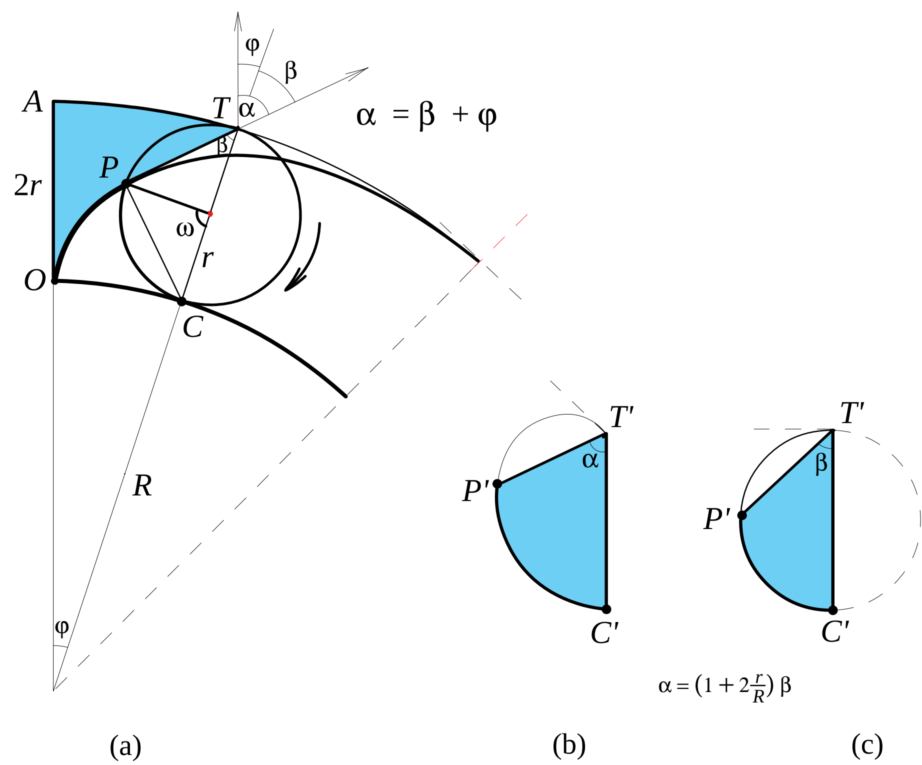

Figure 14: (a) Epicycloidal arc OP. (b) Tangent cluster of tangent sweep $OPTA$ in (a). (c) Area of the tangent cluster in (b) is κ times that of the circular wedge in (c).

result for the length $L(\omega)$ of the epicycloidal arc OP in Figure 14a:

$$L(\omega) = 4r(1 + \frac{r}{R})(1 - \cos\frac{\omega}{2}). \tag{16}$$

For a *hypocycloid*, the circle of radius r rolls inside the circumference of the fixed circle of radius R. In this case $\kappa = 1 - 2r/R$, and the formula for arclength is similar to (16), with the factor $(1 + r/R)$ replaced by $(1 - r/R)$. When $R = \infty$ both reduce to formula (15) for a cycloid. When $\omega = 2\pi$ the rolling disk traces an epicycloid of length $8r(1 + r/R)$ or a hypocycloid of length $8r(1 - r/R)$. These formulas for the arclength of a complete arch are also obtained in [6] by an elementary method.

Note that, according to (7), arclength AT of the free-end curve is $\kappa r\omega$.

General involute and evolute. Consider a family of normals to a given curve σ, as in Figure 15. The envelope τ of these normals is called the *evolute* of σ, and σ, in turn, is called the *involute* of τ. We choose τ as tangency curve and σ as free-end curve. Because $\beta = 0$, the integral in (5) vanishes and we find

$$l(\theta) - l(0) = t(\theta) - t(0). \tag{17}$$

If we take $l(0) = t(0) = 0$, (17) becomes

$$l(\theta) = t(\theta). \tag{18}$$

This verifies the intuitively apparent fact that if a taut string is unwrapped from a point O on τ, its free end will trace the involute σ.

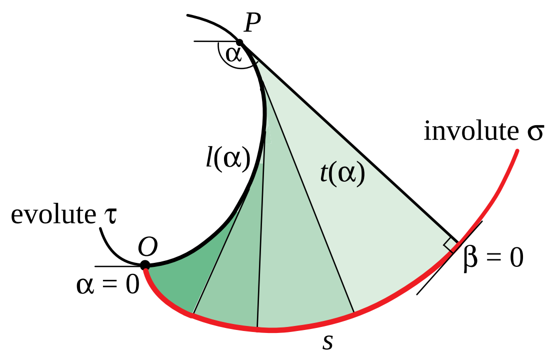

Figure 15: General involute-evolute relations.

From (1) we find that the area $A(\theta)$ of the region between the evolute and involute swept by the tangent from $\alpha = 0$ to $\alpha = \theta$ is

$$A(\theta) = \frac{1}{2}\int_0^\theta l^2(\alpha)d\alpha. \tag{19}$$

Formula (7) for the free-end curve gives the arclength of the involute:

$$s(\theta) = \int_0^\theta l(\alpha)d\alpha. \tag{20}$$

From (20) we can find the arclength l of the evolute τ in terms of s by differentiation:

$$l(\theta) = s'(\theta). \tag{21}$$

Thus, the traditional involute-evolute relations (19), (20), and (21) are merely special cases of our basic relations (1), (7), and (6) when $t(\alpha)$ is chosen to be $l(\alpha)$, the arclength of the tangency curve.

Involute of a circle. Figure 16 shows the special case in which the tangency curve τ is a circle of radius r. Here $t(\alpha) = r\alpha$, where α is measured counterclockwise with $\alpha = 0$ at O. Now use (20) with $l(\alpha) = t(\alpha) = r\alpha$ to obtain

$$s(\theta) = \frac{1}{2} r\theta^2.$$

This intrinsic equation expresses the arclength of the involute of a circle in terms of the unwrapping angle.

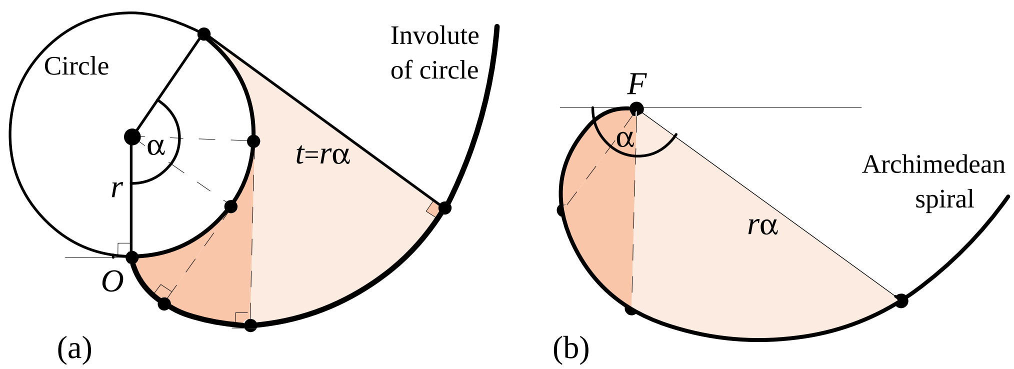

Figure 16: (a) Tangent sweep between a circle and its involute. (b) Its tangent cluster is bounded by an Archimedean spiral.

Figure 16b shows the tangent cluster of the tangent sweep in Figure 16a. Its boundary is an Archimedean spiral because the polar radius is proportional to angle α. Let $A(\theta)$ denote the area of the region swept by the polar radius $r\alpha$ of an Archimedean spiral as α varies from 0 to θ. Formula (19) for the area swept between the circle and its involute implies

$$A(\theta) = \frac{1}{2} \int_0^\theta (r\alpha)^2 \, d\alpha = \frac{r^2\theta^3}{6}. \tag{22}$$

This can be written as $\frac{1}{3}(R^2\theta/2)$, where $R = r\theta$. This result, which was found by Archimedes, states that the area of the region swept by the polar radius of an Archimedean spiral is one-third the area of the circular sector whose radius is the final polar radius of the spiral.

Evolute of a tractrix. Figure 17a shows a tractrix and an arc OP of its evolute. This means that if a string lying along the evolute is unwrapped from O, its free end traces a portion of a tractrix joining point O to point T. The tangent segment from the tractrix at T to the base line AX has constant length k, which is equal to the height of O above AX. We shall determine the length l of arc OP, and the area of the ordinate set $OAXP$ between the evolute and the base AX.

According to (21), $l(\alpha) = s'(\alpha)$, where s is the arclength of the involute, the tractrix in this case, which we have calculated in (12). Renaming l in (12) as s we find, by differentiation, $s'(\alpha) = k \tan \alpha$, so the arclength $l = l(\alpha)$ of the evolute is

$$l(\alpha) = k \tan \alpha. \tag{23}$$

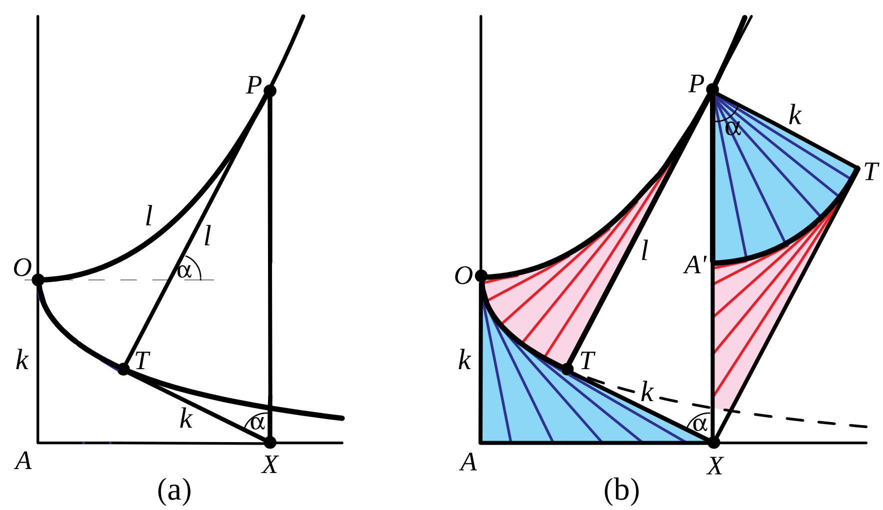

Figure 17: (a) The evolute of a tractrix. (b) Proof that area of ordinate set $OAXP$ is that of rectangle $TPT'X$. This fact implies that the evolute of a tractrix is a catenary.

Next we show that the area of ordinate set $OAXP$ is equal to that of rectangle $TPT'X$ in Figure 17b, with vertical diagonal XP and edges of lengths k and l. Region $OAXP$ consists of three parts: OAX swept by tangent segments of length k to the tractrix, OTP swept by tangent segments of variable length $l(\alpha)$ to the evolute OP, and triangle PTX.

Divide triangle $PT'X$ into two regions, a circular sector $PA'T'$, and region $A'XT'$. Sector $PA'T'$ is the tangent cluster of OAX, the tangent sweep of the tractrix, so they have equal areas. Region $A'XT'$ is swept by tangent segments to the circular arc $A'T'$, which are obtained from tangent sweep OTP by parallel translation of each tangent segment from the evolute. Therefore both swept regions $A'T'X$ and OTP have the same tangent cluster, so they have equal areas.

Thus, the sum of the areas of the two shaded regions on the left of the diagonal PX is equal to the area of the shaded triangle $PT'X$ on the other side of the diagonal. By adding the area of the common unshaded triangle PTX we find

$$\text{area of ordinate set}\, OAXP = \text{area of rectangle}\, TPT'X = kl(\alpha), \tag{24}$$

where $l(\alpha)$ is given by (23).

Catenary as evolute of a tractrix. Now we use our results for arclength and area to deduce the known fact that the evolute of a tractrix is a catenary. Arclength formula (23) and area formula (24) together show that

$$\int_0^x y(u)\, du = k^2 y',$$

which, when differentiated, gives $y = k^2 y''$. The unique solution to this differential equation with $y(0) = k$ and $y'(0) = 0$ is

$$y = k \cosh \frac{x}{k} = k \frac{e^{x/k} + e^{-x/k}}{2}. \tag{25}$$

This is the Cartesian equation for the catenary, and (23) is its intrinsic equation.

Formulas (23) and (24) for arclength and area can also be expressed in terms of hyperbolic functions. Let $L(x)$ denote the arclength of the catenary from $(0,k)$ to (x, y), and let $A(x)$ denote the area of the corresponding ordinate set. Then (23) and (24) give us the classical formulas

$$L(x) = \sinh \frac{x}{k}, \quad \text{and} \quad A(x) = k^2 \sinh \frac{x}{k},$$

where $\sinh x = (e^x - e^{-x})/2$ is the derivative of $\cosh x$.

Also, (24) tells us that the area of the ordinate set between the catenary and the interval AX is equal to the length $l(\alpha)$ of the arc multiplied by the height k of its lowest point above AX, and is also equal to k^2 times the slope of the tangent line at P. Moreover, the ordinate XP of the catenary is equal to $k/\cos\alpha$.

The catenary is well known as the shape of a uniform flexible chain that hangs under its own weight. The standard proof of this fact makes use of a triangle of equilibrium of forces that is similar to triangle PTX in Figure 17a.

Note. We can derive the arclength of the tractrix stated in (12) from the intrinsic equation of the catenary in (23). Choose again the tractrix as the free-end curve with the catenary as tangency curve, and use (7), taking $\beta = 0$ and $t(\alpha) = l(\alpha) = k \tan\alpha$.

Generalized pursuit curve. Figure 18a shows a tangency curve τ and a horizontal free-end curve σ. At a general point of τ a tangent segment of length $t(\alpha)$ cuts off a subtangent of length $b(\alpha)$. For a tractrix, $t(\alpha)$ is constant, and for an exponential, $b(\alpha)$ is constant. Now we consider the more general case in which a convex combination of $t(\alpha)$ and $b(\alpha)$ is constant, say

$$\mu t(\alpha) + \nu b(\alpha) = C, \tag{26}$$

for some choice of nonnegative μ and ν, with $\mu + \nu = 1$. If $\nu = 0$, $\tau(\alpha)$ is constant and the tangency curve τ is a tractrix, which can be regarded as a pursuit curve. If $\mu = \nu$ then $t(\alpha) + b(\alpha)$ is constant, and τ is another pursuit curve in which a fox running on σ is pursued by a dog on τ having the same speed as the fox. Because of these examples, we refer to any curve satisfying (26) as a *generalized pursuit curve*.

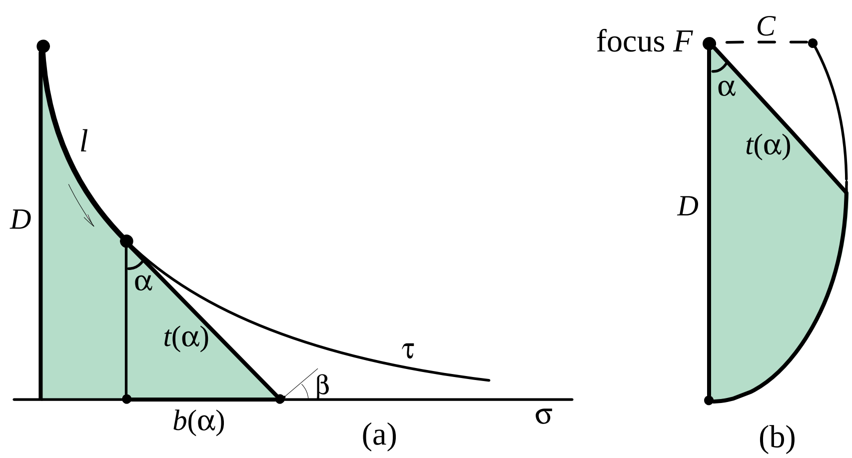

Figure 18: (a) Generalized pursuit curve: $\mu t(\alpha) + \nu b(\alpha) = C$. (b) Tangent cluster of tangent sweep in (a) is a focal sector of a conic section.

Now we will show that the tangent cluster of a generalized pursuit curve is bounded by a conic section with eccentricity ν/μ and a focus at the common point F of the translated segments. An example is shown in Figure 18b.

In Figure 18a we have $\beta = \alpha$ and $b(\alpha) = t(\alpha)\sin\alpha$. Let $D = t(0)$ and let $e = \nu/\mu$, where $\mu \neq 0$. Then $b(0) = 0$ and (26) implies

$$t(\alpha) = \frac{D}{1 + e\sin\alpha}. \tag{27}$$

This is the polar equation with radial distance $t(\alpha)$ of a conic with eccentricity e and focus at F.

Thus, the area of the tangent sweep in Figure 18a is equal to that of the corresponding tangent cluster in Figure 18b, a sector of a conic section swept by a focal radius. This is analogous to the Keplerian sector swept by the radius vector from the sun to an orbiting planet. Note that we have found the area of the shaded region below the pursuit curve without knowing any equation that describes the curve. Its intrinsic equation will be derived below in (30).

As Figure 18a suggests, the distances $t(\alpha)$ and $b(\alpha)$ are asymptotically equal as $\alpha \to \pi/2$. From (26) and (27) we find that the asymptotic pursuit distance is $t(\pi/2) = C = D\mu$.

Now we determine the arclength l of τ from the forward formula (4). Using (27) and $t(0) = D$ in (4) we find

$$l(\theta) = \frac{De\sin\theta}{1 + e\sin\theta} + D\int_0^\theta \frac{\sin\alpha}{(1 + e\sin\alpha)(\cos\alpha)}\,d\alpha. \tag{28}$$

The substitution $u = \sin\alpha$ converts the integral in (28) into

$$\int_0^{\sin\theta} \frac{u}{(1 + eu)(1 - u^2)}\,du.$$

Partial fraction decomposition requires us to consider two cases, $e = 1$ and $e \neq 1$.

Case $e = 1$. In this case the integrand is given by

$$\frac{1}{4}\left(\frac{1}{1+u} - \frac{2}{(1+u)^2} + \frac{1}{1-u}\right),$$

and (28) yields the intrinsic equation for the arclength of a classical pursuit curve:

$$l(\theta) = \frac{D\sin\theta}{2(1 + \sin\theta)} + \frac{D}{4}\log\frac{1 + \sin\theta}{1 - \sin\theta}. \tag{29}$$

In this case, using (7), we find that $s(\theta)$ is equal to $l(\theta)$ as given by (29). This is to be expected because the dog and fox have equal speeds.

Case $e \neq 1$. In this case (28) leads to the following intrinsic equation for the arclength of a generalized pursuit curve:

$$l(\theta) = \frac{De\sin\theta}{1 + e\sin\theta} + \frac{D}{2}\left(\frac{1}{e-1}\log\frac{1 + \sin\theta}{1 + e\sin\theta} - \frac{1}{e+1}\log\frac{1 - \sin\theta}{1 + e\sin\theta}\right). \tag{30}$$

It can be shown (as expected) that (29) is a limiting case of (30) as $e \to 1$.

The exponential curve corresponds to $\mu = 0$ in (26), or the limiting case $e \to \infty$. By taking $l(\theta) - l(\theta_0)$ in (30) and letting $e \to \infty$ in such a way that $D/e \to b$, (where b is the constant subtangent of the exponential) we are led once more to (13). For $e > 1$ the conic (27) is a hyperbola, which, when $e \to \infty$, degenerates into a line at distance b from the focus. This line appears as the dashed vertical line in the cluster triangle in Figure 10, where the common cluster point F is at distance b from this line and serves as the focus of the degenerate hyperbola.

In the generalized pursuit curve (26), σ is the x axis and $\beta = \alpha$. Hence, as mentioned earlier, we can use (7) to calculate $s(\alpha)$ and obtain parametric equations for the rectangular coordinates of any point on the pursuit curve τ.

CONCLUDING REMARKS

The method of sweeping tangents was used in earlier papers [1]-[4] and [7] to calculate areas of many regions bounded by classical curves described by geometric properties rather than by equations. This paper extends the method to determine arclengths of these curves directly from their geometric properties. The method involves two curves, the tangency curve τ and the free-end curve σ, and the tangent vector from one to the other. Intrinsic equations for each of these curves are obtained by expressing their arclengths in terms of the direction angle of the sweeping tangent. We have demonstrated that by judicious choice of τ and σ, these intrinsic equations, in turn, lead to simple straightforward derivations of many known classical results in both polar and rectangular coordinates. In particular, we derived classical involute-evolute relations by choosing the length of the sweeping tangent segment to be the arclength of the tangency curve. Such a special choice could be used to obtain an alternative evaluation of the arclength of a cycloid, an idea introduced by Christopher Wren. The same idea also works for the epicycloid and hypocycloid. There are many other applications of our method not included in this paper.

Brief sketch of proof of Mamikon's theorem

In the most general form of Mamikon's theorem, the tangency curve τ need not lie in a plane. It can be any smooth curve in space, and the tangent sweep will lie on a developable surface, one that can be rolled out flat onto a plane without distortion. The tangent cluster lies on a conical surface whose vertex is the common point F in Figure 3. The general form of Mamikon's theorem states that the area of a tangent sweep to a space curve is equal to the area of its tangent cluster. A detailed proof using differential geometry is given in [2], and a brief sketch of this proof is given here.

Start with a smooth space curve τ described by a position vector $\boldsymbol{X}(l)$, where l, the arclength function for the curve, varies over an interval, say $0 \le a \le l \le b$. The unit tangent vector to τ is the derivative $d\boldsymbol{X}/dl$, which we denote by $\boldsymbol{T}(l)$. The derivative of the unit tangent is given by

$$\frac{d\boldsymbol{T}}{dl} = \kappa(l)\boldsymbol{N}(l),$$

where $\boldsymbol{N}(l)$ is the principal unit normal and $\kappa(l)$ is the curvature.

Curve τ generates a surface S that can be represented by the vector parametric equation

$$\boldsymbol{y}(l,u) = \boldsymbol{X}(l) + u\boldsymbol{T}(l),$$

where u varies over an interval whose length can vary with l, say $0 \le u \le f(l)$. As the pair of parameters $(u,\, l)$ varies over the ordinate set of the function f over the interval $[a,\, b]$, the surface S is swept out by tangent segments extending from the initial curve τ to another curve σ described by the position vector $\boldsymbol{y}(l,\, f(l))$.

Geometrically, S is a developable surface, that is, it can be rolled out flat on a plane without distortion. We refer to surface S generated from curve τ in this fashion as a tangent sweep. The area of S is given by the double integral

$$a(S) = \int_a^b \int_0^{f(l)} || \frac{\partial \boldsymbol{y}}{\partial l} \times \frac{\partial \boldsymbol{y}}{\partial u} ||\, du\, dl.$$

A straightforward calculation of the integrand shows that

$$a(S) = \int_a^b (\int_0^{f(l)} u\, du)\, \kappa(l)\, dl = \frac{1}{2} \int_a^b f^2(l)\kappa(l)\, dl.$$

Next, imagine the arclength l expressed as a function of the angle α between the tangent vector $\boldsymbol{T}$ and a fixed tangent line, say the tangent line corresponding to $l = a$. When l is expressed in terms of α, the function $f(l)$ becomes a function of α, and we write $f(l) = t(\alpha)$, the length of the sweeping tangent vector. On the surface S, α is the angle between tangent geodesics, so the curvature κ is the rate of change of α with respect to arclength, $\kappa = d\alpha/dl$. In the last integral we make a change of variable expressing l as a function of α. Then $f^2(l) = t^2(\alpha)$, $\kappa(l)dl = d\alpha$, and the integral for $a(S)$ becomes

$$a(S) = \frac{1}{2} \int_{\alpha_1}^{\alpha_2} t^2(\alpha)\, d\alpha, \tag{31}$$

where α_1 and α_2 are the initial and final angles of inclination corresponding to $l = a$ and $l = b$, respectively. Formula (31) shows that area $a(S)$ does not depend explicitly on the arclength of τ; it depends only on the angles α_1 and α_2. In fact, $a(S)$ is equal to the area of a plane radial set with polar coordinates (t, α) satisfying $0 \le t \le t(\alpha)$ and $\alpha_1 < \alpha \le \alpha_2$. When τ is a plane curve, (31) becomes (1) and gives the planar form of Mamikon's theorem.

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