

Finding Centroids the Easy Way

Introduction

Archimedes (287–212 B.C.), often called the greatest mathematician of ancient times, created the concept of centroid, which he used in many of his writings on mechanics. But we can only speculate on what he had in mind when he referred to centroids because none of his extant works provides an explicit definition of the concept.

The centroid, for us, is a special case of a more general concept called *center of mass*, which plays an important role in physics. Newton’s laws of mechanics are usually stated so they apply to a point particle described by its position, velocity, and acceleration. But every object we see in nature is really a compound body made up of smaller parts, ultimately of atoms, and physicists apply Newton’s laws to such extended objects. They do this by imagining the body replaced by a single point where all its mass is concentrated. This point is called the center of mass. In uniform symmetric bodies it is identified with the center of symmetry. The position, velocity, and acceleration of a compound body are really those of the center of mass.

The mathematical definition of center of mass for two mass points m_1 and m_2 located at points x_1 and x_2 on the x -axis is

$$\bar{x} = \frac{m_1x_1 + m_2x_2}{M} \quad (1)$$

where $M = m_1 + m_2$.

One can think of \bar{x} as a weighted average, in which each position x_i is weighted by the fraction m_i/M . If $m_1 = m_2$ both fractions are $1/2$ and the center of mass is halfway between x_1 and x_2 , as one would expect from symmetry.

The definition is readily generalized to any finite number of mass points m_1, \dots, m_n of total mass $M = m_1 + \dots + m_n$ by using the weight m_i/M at each point x_i . If the masses are on a line at points x_1, \dots, x_n , the center of mass is the point

$$\bar{x} = \frac{m_1x_1 + \dots + m_nx_n}{M}. \quad (2)$$

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If the mass points are not necessarily on a line but are in a plane or in space at positions described by n vectors $\mathbf{r}_1, \dots, \mathbf{r}_n$, definition (2) is replaced by

$$\bar{\mathbf{r}} = \frac{m_1\mathbf{r}_1 + \dots + m_n\mathbf{r}_n}{M}. \quad (3)$$

Two important properties of center of mass follow at once from this definition. Suppose the position vectors $\mathbf{r}_1, \dots, \mathbf{r}_n$ are drawn from the origin O of some coordinate system. If the origin is translated to a new position by a given vector \mathbf{a} , each position vector \mathbf{r}_k is replaced by $\mathbf{r}_k - \mathbf{a}$ so the center of mass relative to the new origin is equal to

$$\frac{m_1(\mathbf{r}_1 - \mathbf{a}) + \dots + m_n(\mathbf{r}_n - \mathbf{a})}{M} = \frac{m_1\mathbf{r}_1 + \dots + m_n\mathbf{r}_n}{M} - \mathbf{a} = \bar{\mathbf{r}} - \mathbf{a}.$$

In other words, the center of mass is shifted by the same amount as each position vector. So it is an intrinsic property of the body that does not depend on the origin of the coordinate system.

Next, if each position vector \mathbf{r}_k is expanded or contracted by the same scaling factor t , then the center of mass $\bar{\mathbf{r}}$ gets multiplied by the same factor t . We call this the scaling property of center of mass.

Incidentally, the center of mass can be adapted to define the centroid of a number of points $\mathbf{r}_1, \dots, \mathbf{r}_n$. This is simply the center of mass defined by (3) when all the masses are equal. In this case $m_i = M/n$ for each i and equation (3) becomes

$$\bar{\mathbf{r}} = \frac{\mathbf{r}_1 + \dots + \mathbf{r}_n}{n}. \quad (4)$$

Mathematically, this is equivalent to assigning weight 1 to each point.

When we deal with a system whose total mass is distributed along all the points of an interval, or along a curve, or throughout some region in the plane or in space rather than at a finite number of discrete points, the concepts of mass and center of mass are defined by integrals rather than sums. These definitions can be found in most calculus textbooks. For uniform bodies the center of mass is called the *centroid*.

This article determines the centroid of several plane figures without using the formal machinery of integral calculus. Examples include the vertices of a triangle, an L-shaped

lamina (a *lamina* can be thought of as a plane figure cut out of a thin piece of cardboard or plywood of uniform thickness), the boundary of a triangle, a triangular lamina, and a circular arc. For a lamina with a center of symmetry, the centroid is at the center. To find the centroid of nonsymmetric laminas we shall use some basic properties of centroids, described below, together with a more profound property that we call the Archimedes Lemma because it resembles Proposition 4 in Book I of Archimedes's *On the equilibrium of planes*.

The Archimedes Lemma

Archimedes Lemma. *If an object is divided into two smaller objects, the center of mass of the compound object lies on the line segment joining the centers of mass of the two smaller objects.*

For objects made up of a finite number of discrete masses a simple proof follows at once from the definition in (3), and the same type of proof works for more general mass distributions defined by integrals. We don't know how Archimedes proved Proposition 4 because he alludes to a previously proved property in his lost treatise *On levers*.

Example 1. Centroid of the vertices of any triangle.

Figure 1(a) shows a system of three points, the vertices of a triangle. We will show that the centroid of this system is at the intersection of the three medians of the triangle. (A median is the line segment from a vertex to the midpoint of the opposite side.) First, consider a pair of vertices as one object, whose centroid is midway between these vertices, and the opposite vertex as the second object, and apply the Archimedes Lemma. This places the centroid c of the combined object on a median. Repeat the argument for each pair of vertices and we see that c is on all three medians. Therefore the three medians of any triangle intersect at a point, which is the centroid of the vertices. Later we will show that the centroid of the vertices is also the centroid of the entire triangular lamina.

To find the distance from each vertex to the centroid, refer to the parallelogram formed by the dotted lines in Figure 1(b). Its upper edge joins two midpoints of the original triangle. Its lower edge joins midpoints of the inner triangle having two vertices at the original base and one vertex at the centroid. Two medians pass through the diagonals of this parallelogram. The diagonals bisect each other, so the inter-

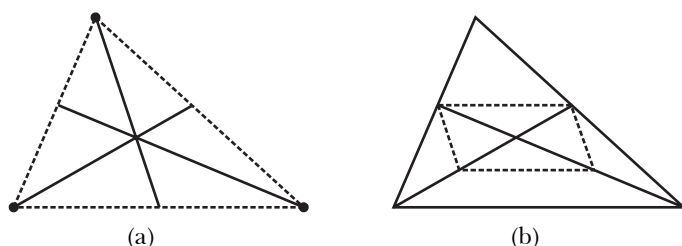


Figure 1. The centroid of the vertices of any triangle is at the intersection of the medians.

section point of the two medians trisects both of them. In other words, the distance from each vertex to the centroid of the vertices is two-thirds the length of the median.

In general, the Archimedes Lemma involves three objects A, B, C whose centers of mass a, b, c lie on a line. But a line is uniquely determined by any two of its points. Therefore, if we know the center of mass c of a compound object C and the center of mass b of one of its parts B , then the center of mass a of the other part A is on the line joining b and c . This variation of the Archimedes Lemma vastly increases the scope of its applicability, as we will demonstrate later with a number of examples involving plane laminas. Besides the Archimedes Lemma we will use several simple properties of centroids, each of which seems intuitively evident when the centroid of a plane lamina is regarded as the point about which the gravitational forces acting on the lamina are balanced.

Fundamental Properties of Centroids of Plane Laminas

- 1. Translation property.** Translation of a lamina by a given vector translates its centroid to the centroid of the translated lamina.
- 2. Rotation property.** Rotation of a lamina through a given angle about an axis perpendicular to the plane of the lamina rotates its centroid to the centroid of the rotated lamina.
- 3. Centroid of a symmetric object.** If a lamina has a center of symmetry, the centroid of the lamina is at the center of symmetry.
- 4. Centroid of a lamina with an axis of symmetry.** If a lamina has an axis of symmetry, its centroid lies somewhere on this axis.
- 5. Centroid of an object formed from a lamina and any of its translates.** A lamina together with any of its translates is a compound object whose centroid is at the midpoint of the line segment joining the centroids of the two laminas.

It is easy to verify these properties for objects consisting of a finite number of discrete points in a plane, using equation (4). Similar proofs can be given for the more general case where centroids are defined by integrals. We will give proofs only for properties (2) and (5) for objects consisting of a finite number of points.

Proof of (2). For this proof we represent each point \mathbf{r}_k as a complex number. By property (1) there is no loss of generality if we assume the rotation axis passes through the origin, in which case rotation of \mathbf{r}_k through an angle θ is the same as multiplying \mathbf{r}_k by $e^{i\theta}$, so the centroid of the rotated system is equal to

$$\frac{\mathbf{r}_1 e^{i\theta} + \cdots + \mathbf{r}_n e^{i\theta}}{n} = \overline{\mathbf{r}} e^{i\theta}.$$

Proof of (5). If the first object has points $\mathbf{r}_1, \dots, \mathbf{r}_n$ with centroid $\bar{\mathbf{r}}$ then the translated object has points $\mathbf{r}_1 + \mathbf{a}, \dots, \mathbf{r}_n + \mathbf{a}$, where \mathbf{a} is the translation vector. The composite object consists of $2n$ points whose centroid $\bar{\mathbf{R}}$ is given by

$$\begin{aligned}\bar{\mathbf{R}} &= \frac{\mathbf{r}_1 + \dots + \mathbf{r}_n + (\mathbf{r}_1 + \mathbf{a}) + \dots + (\mathbf{r}_n + \mathbf{a})}{2n} \\ &= \frac{\mathbf{r}_1 + \dots + \mathbf{r}_n}{n} + \frac{1}{2}\mathbf{a} = \bar{\mathbf{r}} + \frac{1}{2}\mathbf{a},\end{aligned}$$

so $\bar{\mathbf{R}}$ is at the midpoint of the segment joining $\bar{\mathbf{r}}$ and $\bar{\mathbf{r}} + \mathbf{a}$.

Applications

We have already used the Archimedes Lemma to find the centroid of the vertices of a triangle. We use it again to find the centroids of a variety of nonsymmetric objects. All the examples considered below are assumed to be uniform bodies for which the center of mass and the centroid are identical. As mentioned above, the centroid of a uniform symmetric object, such as a rectangular lamina or a parallelogram, is at the center of symmetry.

Example 2. L-shaped lamina.

Figures 2(a) and 2(b) show an L-shaped plane lamina decomposed in two different ways into rectangular pieces. In each case, the centroid of the lamina lies on the line segment joining the centroids of the two rectangular pieces. Consequently, the centroid of L is at the point of intersection of these line segments, as shown in Figure 2(c).

In actual practice there may be some difficulty in determining the point of intersection of the two line segments in Figures 2(a) and 2(b) because they could be nearly parallel.

Figure 3 shows another way to find this point. Enlarge the L-shaped lamina by the dotted lines as shown to form a large rectangle whose centroid is at its center. The smaller rectangle in the corner has its centroid at its center. Therefore, by the variation of the Archimedes Lemma, the centroid of L must also lie on the line joining these two centers. This line intersects each of the lines in Figures 2(a)

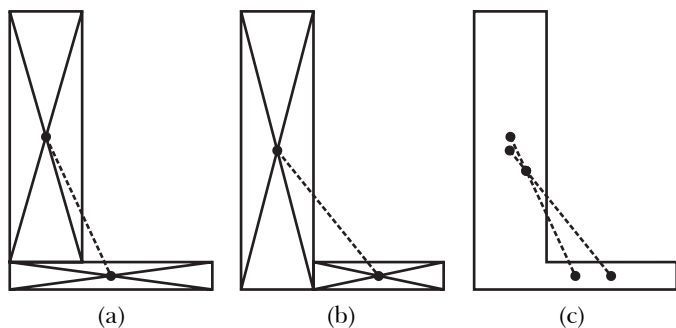


Figure 2. Centroid of an L-shaped lamina found by using Archimedes's Lemma twice.

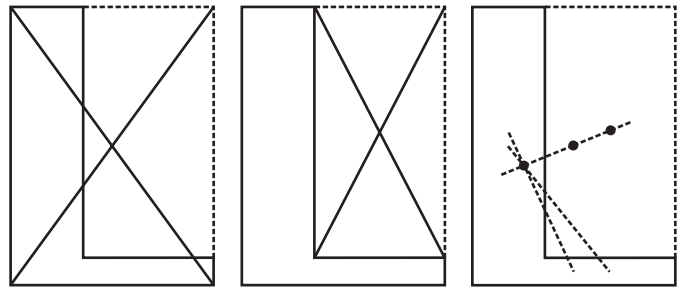


Figure 3. Alternate method for finding the centroid of an L-shaped lamina.

and 2(b) at the required centroid of L .

Example 3. Two intersecting line segments (a limiting case of Example 2).

When each rectangle of the L-shaped lamina in Example 2 degenerates to a line segment, the object consists of two perpendicular line segments. We can easily find the centroid of such an object even if the segments are not perpendicular. Figure 4(a) shows a body APB consisting of two legs PA and PB . Its centroid lies on the segment joining the midpoints M and N of the legs. If the legs have equal length the figure is symmetric and its centroid is halfway between M and N . If the lengths are unequal, let PA be the shorter leg and divide body APB into two parts, a symmetric part APA' , where PA' has length PA , plus a residual segment $A'B$. The centroid of $A'B$ is at its midpoint p , and the symmetric part APA' has its centroid at the point s midway between the centers M and M' of the equal sides, as shown in Figure 4(b). By the Archimedes Lemma, the centroids p and s lie on the line that contains the centroid c of the compound body APB . This line intersects segment MN at c .

Example 4. An alternate method for finding the centroid in Example 3.

An alternative method for determining the centroid c leads to a surprising and useful connection with the triangle MON whose vertices are the midpoints of the sides of triangle APB , shown in Figure 5. Instead of dividing APB into a symmetric part plus an extra segment as was done in Example 3, we extend the shorter leg AP to point Q so that AQ has the same

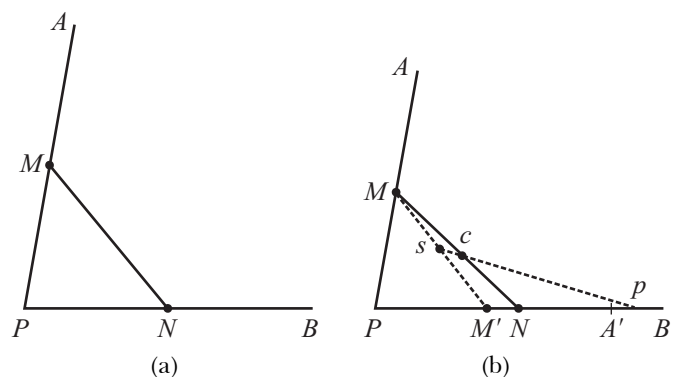


Figure 4. Determining the centroid of two intersecting line segments.

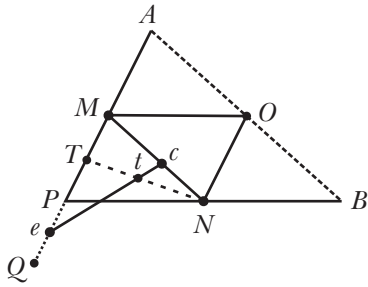


Figure 5. Alternative method for finding the centroid of APB .

length as PB . The composite body consisting of segments AQ and PB of equal length has its centroid at point t , halfway between the midpoints T of AQ and N of PB . Now let e denote the midpoint of PQ . By the variation of the Archimedes Lemma, the centroid c of APB lies on the line joining the centroids e and t . This line intersects the segment MN at c .

Example 5. Yet another method for finding the centroid of Example 4.

Now we show that the line through e , t , and c in Figure 5 also passes through the midpoint O of AB . Quadrilateral $PNOM$ is a parallelogram because MO is parallel to PB and ON is parallel to AP . First we show that line eO bisects angle MON , then we show that line eO also contains t and c .

Draw a line through e parallel to PN and extend ON until it intersects this line at R , say, as shown in Figure 6. We get a new parallelogram $eROM$ which is also a rhombus (all four of its sides are equal). The reason for this is that $eR = PN = \frac{1}{2}PB = \frac{1}{2}AQ$, and $Me = MP + Pe = \frac{1}{2}AP + \frac{1}{2}PQ = \frac{1}{2}AQ = eR$. Hence eO is a diagonal of this rhombus so it bisects angle MON . But segment TN divides the rhombus into two congruent quadrilaterals $MONT$ and $eRNT$, so t , the midpoint of TN , is at the center of the rhombus. Hence diagonal eO passes through t and hence also through c because e , t , and c lie on a line. Therefore, we have another way to find the centroid c that does not require extending the leg AP . Complete the triangle having PA and PB as two edges, and draw the triangle MON joining the midpoints of its sides. The intersection of MN and the bisector of angle MON is the centroid c .

Example 6. The centroid of the boundary of a triangle.

The boundary of a triangle can be regarded as an object made

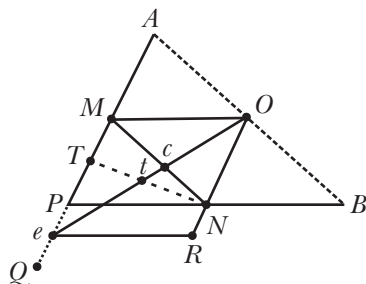


Figure 6. The centroid c lies on MN and also on the bisector of angle MON .

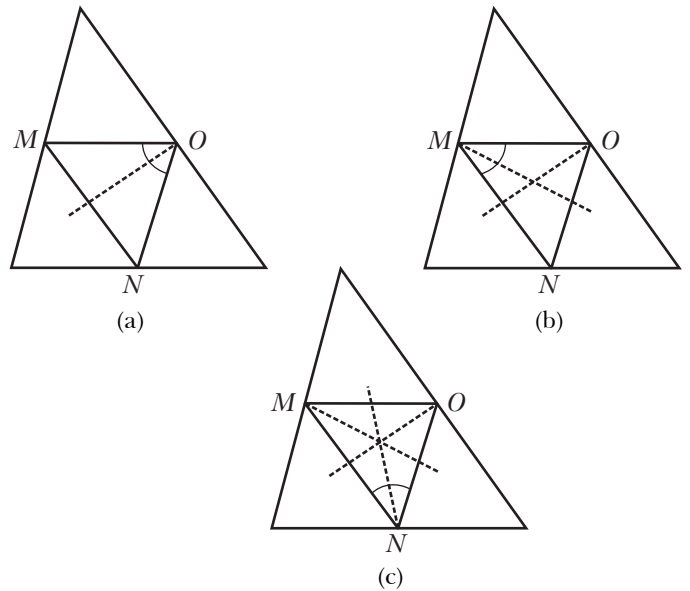


Figure 7. The centroid of the boundary of a triangle is at the intersection of three angle bisectors of the medial triangle.

from a uniform wire. Its centroid can be determined by applying the Archimedes Lemma twice. First imagine the boundary of the triangle as a composite body consisting of two line segments meeting at one of the vertices plus a second body consisting of the edge opposite this vertex. This can be done in three ways, as illustrated in Figures 7(a), (b), and (c). In each case, the centroid of the first body can be found as in Example 5, and the centroid of the second body lies on the angle bisector joining these two, shown as a dashed line in each of Figures 7(a), (b), (c). These three angle bisectors will intersect at the centroid of the triangular boundary.

Example 7. The centroid of a triangular lamina.

Archimedes was the first to determine the centroid of a triangular lamina T (T includes the boundary of the triangle and all points inside). He found that the centroid of T is at the point of intersection of the three medians. In fact, Archimedes gives two proofs, both using contradiction. However, both proofs are complicated, involving a large number of auxiliary construction lines, and the proofs are not easy to follow.

We give a simple direct proof. Dissect T into four similar triangles by its midlines, as shown in Figure 8(a). The four

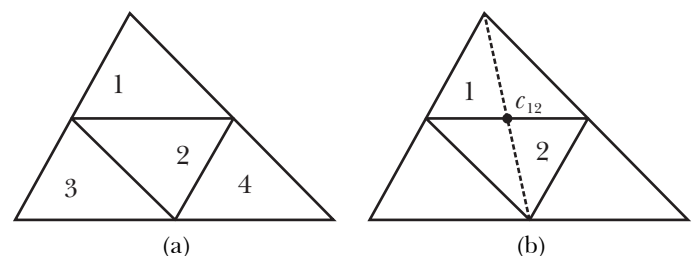


Figure 8. A triangle divided by its midlines into four congruent triangles.

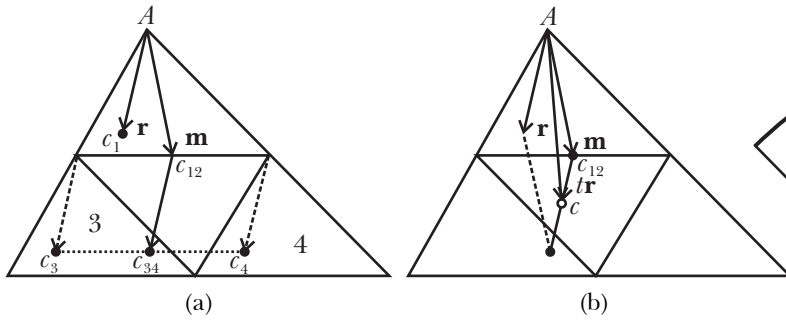


Figure 9. Determining the centroid of a triangular lamina.

pieces are congruent and each edge is half that of the corresponding edge of T . The pieces numbered 3 and 4 are translates of triangle 1, whereas piece number 2 in the center is obtained by rotating triangle 1 through 180° about the center of their common edge.

Triangles 1 and 2 together form a parallelogram whose centroid c_{12} is at its center, which is at the midpoint of the common base of triangles 1 and 2 and also on the median of both triangles 1 and T , as shown in Figure 8(b).

Now refer to Figure 9. Let \mathbf{r} be a vector from the uppermost vertex A of T to a point alleged to be the centroid c_1 of triangle 1, and let \mathbf{m} be the vector (along the median) from A to the midpoint of the opposite side of triangle 1. We wish to prove that \mathbf{r} and \mathbf{m} are parallel. By the translation property of centroids, the terminal points of the dotted vectors in Figure 9(a) represent the centroids c_3 and c_4 of triangles 3 and 4, respectively. Since triangle 4 is a translation of triangle 3, property 5 tells us that the centroid c_{34} of triangles 3 and 4 together lies midway between c_3 and c_4 . Now the large triangle T is divided into two parts, the parallelogram with centroid c_{12} and the two triangles 3 and 4 with centroid c_{34} , so by the Archimedes Lemma the centroid c of T is on the line segment joining c_{12} and c_{34} which, in turn, is parallel to and equal in length to the vector \mathbf{r} . Hence the vector from c_{12} to c is $t\mathbf{r}$, where t lies between 0 and 1, as shown in Figure 9(b). Therefore the vector from A to c is the vector sum $\mathbf{m} + t\mathbf{r}$. But triangle 1 can be obtained from T by scaling all distances from vertex A by the factor $\frac{1}{2}$, so by the scaling property of centroids we have $\mathbf{r} = \frac{1}{2}(\mathbf{m} + t\mathbf{r})$, which implies $\mathbf{m} = (2 - t)\mathbf{r}$. This shows that \mathbf{r} and \mathbf{m} are parallel because \mathbf{m} is a positive scalar times \mathbf{r} .

Hence the two centroids c_1 and c are on the median from vertex A to the opposite side. Applying the same argument to each of the other vertices we see that c is on each of the three medians. This shows that the medians of any triangle intersect at the centroid of the triangle. As already shown in Example 1, this point is also the centroid of the three vertices, and the distance from each vertex to the centroid is two-thirds the length of the median.

Note. The authors have devised a variation of this proof that the reader may wish to discover independently. It involves dividing the large triangle into nine congruent triangles by trisecting each of its edges. The corner triangles

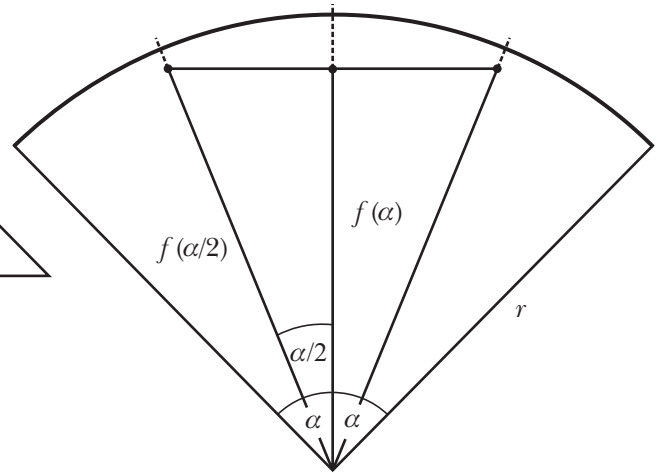


Figure 10. Denote by $f(\alpha)$ the distance from the center of the circle to the centroid of the arc subtended by an angle of 2α radians.

are translates of each other. The lamina is made up of the three corner triangles, together with a centrally symmetric hexagon such that each pair of opposite edges are of equal length. One fringe benefit of this modified proof is that it not only locates the centroid but at the same time gives the distance of the centroid from each vertex.

Example 8. The centroid of a circular arc.

The Archimedes Lemma can also be used to find centroids of curved figures. For example, a standard calculus exercise is to show that the centroid of a circular arc of radius r that subtends a central angle of 2α radians lies along the radius that bisects the arc at a distance $r(\sin\alpha)/\alpha$ from the center. When $\alpha = 0$ this distance is to be interpreted as r , which agrees with the fact that $(\sin\alpha)/\alpha \rightarrow 1$ as $\alpha \rightarrow 0$.

This result can be deduced from the basic properties of centroids. Refer to Figure 10, which shows a circular arc of radius r with central angle 2α radians. By symmetry, the centroid lies on the radial line that bisects this arc. Let $f(\alpha)$ denote the distance from the center of the circle to the centroid. The bisector divides the arc into two symmetric arcs of α radians, and the two centroids of these smaller arcs are located at a distance $f(\alpha/2)$ from the center. By the Archimedes Lemma, the centroid of the larger arc 2α must lie on the line segment joining the centroids of the smaller arcs. This line segment is perpendicular to the radial line from the center to the centroid of the larger arc. Therefore, from the right triangle in the figure we see that $f(\alpha) = f(\alpha/2)\cos(\alpha/2)$ or, replacing α by 2α , we find that

$$f(2\alpha) = f(\alpha)\cos\alpha, \text{ with } f(0) = r. \quad (5)$$

This is a functional equation relating $f(2\alpha)$ with $f(\alpha)$. We will show that the only function that satisfies (5) and is continuous at 0 is $f(\alpha) = r(\sin\alpha)/\alpha$, as required.

Solution of the functional equation (5). The function f_0 defined by $f_0(\alpha) = r(\sin\alpha)/\alpha$ if $\alpha \neq 0$, $f_0(0) = r$, satisfies (5) because of the double-angle formula $\sin(2\alpha) = 2\sin\alpha\cos\alpha$. Also, f_0 is continuous at 0. Now let f be any solution of (5)

that is continuous at 0, and define $g(\alpha) = f(\alpha)/f_0(\alpha)$. We wish to prove that g is the constant function 1. Applying the functional equation we find

$$\begin{aligned} g(2\alpha) &= \frac{f(2\alpha)}{f_0(2\alpha)} = \frac{f(\alpha)\cos(\alpha)}{r\sin(2\alpha)/2\alpha} \\ &= \frac{f(\alpha)\cos(\alpha)2\alpha}{2r\sin(\alpha)\cos(\alpha)} = \frac{f(\alpha)\alpha}{r\sin(\alpha)} = g(\alpha). \end{aligned}$$

Hence, $g(2^n\alpha) = g(\alpha)$ for $n = 1, 2, \dots$. If $x = 2^n\alpha$ this last relation states that $g(x) = g(x/2^n)$ for $n = 1, 2, \dots$. Letting $n \rightarrow \infty$ and invoking continuity at 0 we find $g(x) = g(0) = 1$ for all x , hence $f(\alpha) = f_0(\alpha)$, as asserted.

Note. A similar argument shows that the centroid of a circular sector of radius r subtending a central angle 2α lies on the radial line that bisects the sector at a distance $\frac{2}{3}r(\sin\alpha)/\alpha$ from the center. The reader may wish to deduce this as an exercise using the method of Example 8. The Archimedes Lemma leads to the same functional equation as above, but in this case the limiting value as $\alpha \rightarrow 0$ is $f(0) = \frac{2}{3}r$. In fact, the method of Example 8 works in some cases of non-uniform mass distribution with radial symmetry, the centroidal distance being $cr(\sin\alpha)/\alpha$ where c is a constant depending on the mass distribution. ■