TOM M. APOSTOL and MAMIKON A. MNATSAKANIAN

Cycloidal Areas without Calculus

1. Introduction

For centuries mathematicians have been interested in curves that can be constructed by simple mechanical instruments. Among these curves are various cycloids used by Apollonius around 200 B.C. and by Ptolemy around 200 A.D. to describe the apparent motions of planets. The simplest cycloid is the curve traced out by a point on the circumference of a circular disk that rolls without slipping along a horizontal line; it forms a sequence of arches resting on the line, as shown in Figure 1.

Let S denote the area of the region above the line and below one of these arches (shown shaded in Figure 1). A routine use of integral calculus reveals that S is three times the area of the rolling circular disk, which we express symbolically as follows:

$$S = 3 \times \odot. \tag{1}$$

The derivation of this formula using integral calculus requires parametric or Cartesian equations for the cycloid.

This paper solves the more general problem in which the rolling circle is replaced by any regular polygon. The result is obtained by a geometrical method, and the area formula for the cycloid is obtained as a limiting case. We use the formula for the area of a circular sector, but there is no need to know the equations representing the cycloid.

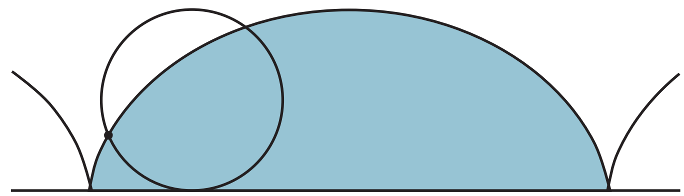

Figure 1. *Cycloid traced out by a point on the circumference of a rolling circle.*

2. Cyclogons

When a regular polygon rolls without slipping along a straight line, a given vertex on its circumference traces out a curve we call a *cyclogon*. Like the cycloid, a cyclogon consists of a sequence of arches resting on the line, as shown by the example of a rolling pentagon in Figure 2. Each arch, in turn, is composed of circular arcs, equal in number to one fewer than the number of vertices of the polygon. The arcs need not have the same radius.

If S denotes the area of the region above the line and below one of these arches, we will show that, in place of (1), we have the elegant and surprising result

$$S = \otimes + 2 \times \odot, \tag{2}$$

where $\otimes$ denotes the area of the rolling polygon and $\odot$ is the area of the disk that circumscribes the polygon. The circle can be regarded as the limiting case obtained by letting the number of edges increase without bound in a regular polygon. Similarly, the cycloid is the limiting case of a cyclogon. Equation (1) for the area of the region under one arch of a cycloid is now revealed as a limiting case of Equation (2).

We begin with two simple examples, a rolling triangle, and a rolling square.

3. Rolling equilateral triangle

Figure 3 shows one arch of a cyclogon traced out by a rolling equilateral triangle whose edges have length a. The region under this arch and above the line consists of two equal circular sectors of radius a and one equilateral triangle. Each circular sector has area $(\pi/3)a^2$ which is also the area of the circular disk that circumscribes an equilateral triangle of edge-length

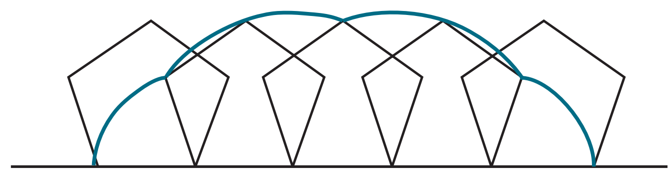

Figure 2. *Cyclogon traced out by a vertex on the boundary of a rolling regular pentagon.*

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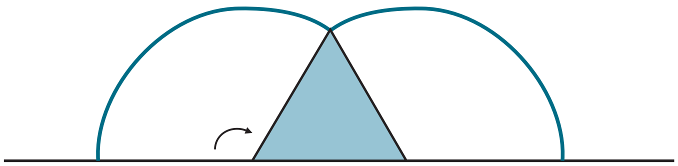

Figure 3. *One arch of a cyclogon traced out by a rolling equilateral triangle.*

a. Therefore

$$S = \text{area of } \Delta + 2 \times \frac{\pi}{3} a^2 = \text{area of } \Delta + 2 \times \odot,$$

which proves (2) in this case.

4. Rolling square

Figure 4 shows a cyclogon traced out by a rolling square. The region under this arch consists of two right triangles plus three circular quadrants, two of radius a (the edge-length of the square), and one of radius $a\sqrt{2}$ (the diagonal of the square). The two right triangles have total area a^2, the area of the rolling square, and the total area of the three circular quadrants is

$$2 \times \frac{\pi}{4} a^2 + \frac{\pi}{4}\left(a\sqrt{2}\right)^2 = 2 \times \pi \left(a \frac{\sqrt{2}}{2}\right)^2 = 2 \times \odot.$$

Therefore we have $S = a^2 + 2 \times \odot$, which proves (2) in this case as well.

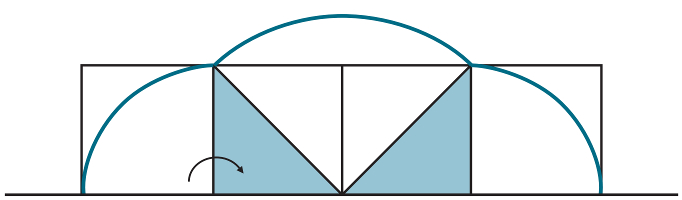

Figure 4. *One arch of a cyclogon traced out by a rolling square.*

5. Rolling *n*-gon

In the general case of a regular polygon with n vertices, the region under one arch of the cyclogon consists of $n-2$ triangles and $n-1$ circular sectors, each subtending an angle of $2\pi/n$ radians.

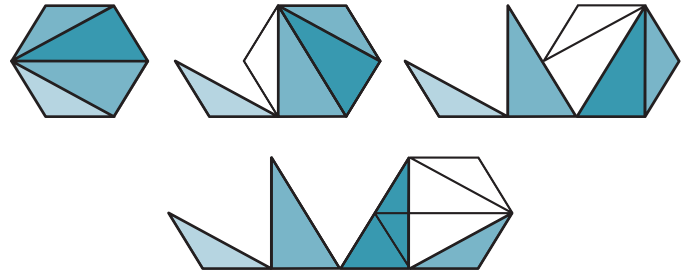

Figure 6. *Triangular dissection of a regular hexagon and distribution of its footprints.*

The $n-2$ triangles can be regarded as "footprints" left by the triangular pieces obtained by dissecting the original polygon with $n-3$ diagonals from a given vertex to each of the nonadjacent vertices, as illustrated in Figure 5 for $n = 6$. The sum of the areas of these triangles is equal to the area of the region enclosed by the regular polygon. This is illustrated for the regular hexagon in Figure 6.

The radii of the circular sectors are the lengths of the segments from one vertex to each of the remaining $n-1$ vertices. A sector of radius r_k subtending an angle of $2\pi/n$ radians has area $\pi r_k^2 / n$, so the sum of the areas of the $n-1$ sectors is equal to

$$\frac{\pi}{n} \sum_{k=1}^{n-1} r_k^2.$$

In the next section we will show that the sum of the squares of these radii is equal to $2nR^2$, where R is the radius of the circle that circumscribes the polygon. Therefore the sum of the areas of the sectors is equal to $2\pi R^2$, which is twice the area of the circumscribing disk. In other words, (2) is a consequence of the relation

$$(3) \qquad \sum_{k=1}^{n-1} r_k^2 = 2nR^2.$$

6. An extension of the Pythagorean Theorem for regular polygons

The result in (3), which is needed to calculate the sum of the areas of the circular sectors in the foregoing section, is of

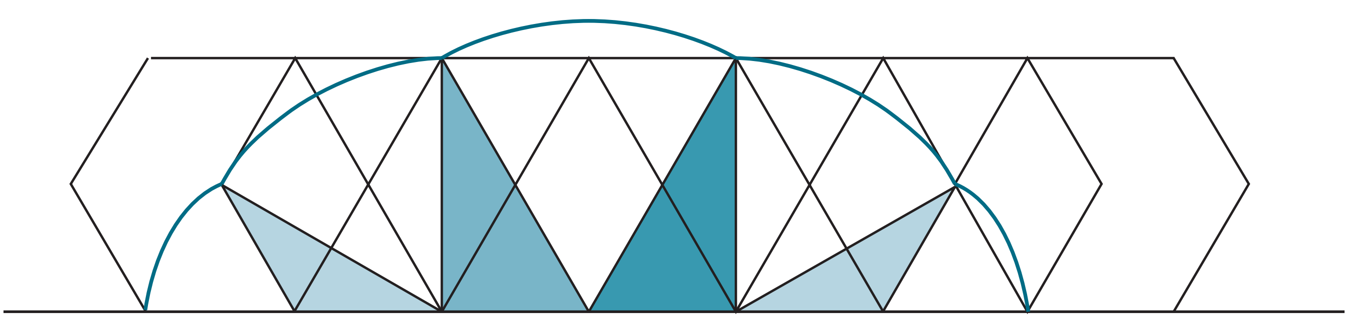

Figure 5. *"Footprints" left by triangular pieces of a rolling hexagon.*

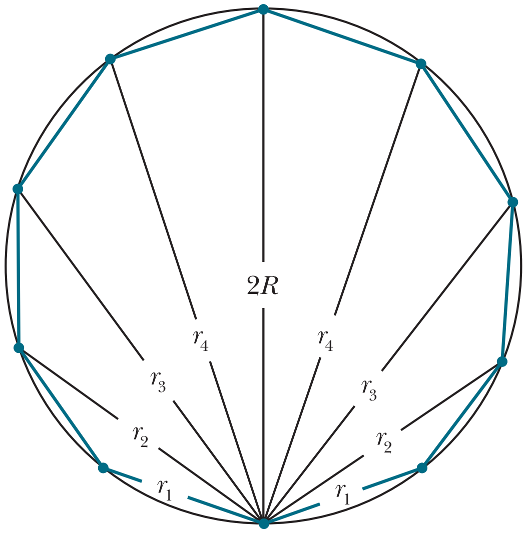

Figure 7. *Nine segments drawn from one vertex of a regular decagon to the other nine.*

independent interest because it reduces to the Pythagorean theorem when the regular polygon is a square. The authors have not been able to locate this surprising theorem in any published work, so it may be new.

Theorem. *The sum of the squares of the* $n - 1$ *segments drawn from one vertex of a regular n-gon to the remaining vertices is equal to* $2nR^2$*, where R is the radius of the circumscribing circle.*

7. Proof for regular polygons with an even number of sides

The proof for even n makes repeated use of the Pythagorean Theorem. It is illustrated for the case $n = 10$ in Figure 7, which shows nine segments drawn from one vertex of a regular decagon to the other nine vertices.

In Figure 7 there are four segments, labeled as r_1, r_2, r_3, r_4, and four mirror images, plus the diameter of length $2R$, so the sum in question is

$$2\sum_{k=1}^{4} r_k^2 + 4R^2. \tag{4}$$

Figure 8 shows two segments from the opposite extremity of the diameter to consecutive vertices. By symmetry with respect to a horizontal diameter, these segments have lengths r_1 and r_2. The new segment r_1 meets the old segment r_4 on the circle and, together with the diameter, forms a right triangle with hypotenuse $2R$. (Here we use the fact that any triangle inscribed in a semicircle is a right triangle with the diameter as hypotenuse.) Applying the Pythagorean Theorem to this right triangle we find

$$r_1^2 + r_4^2 = 4R^2. \tag{5}$$

Similarly, the new segment r_2 intersects the old segment r_3 and forms another right triangle with hypotenuse $2R$. Applying the Pythagorean Theorem once more we find

$$r_2^2 + r_3^2 = 4R^2 \tag{6}$$

so the sum in (4) is equal to

$$2\sum_{k=1}^{4} r_k^2 + 4R^2 = 16R^2 + 4R^2 = 20R^2$$

which proves the Theorem for $n = 10$.

In the general case of even n, one of the $n - 1$ segments is the diameter $2R$ of the circumscribing circle, and the other $n - 2$ segments form $(n - 2)/2$ pairs symmetrically located with respect to the diameter. The same argument just given for the case $n = 10$ shows that

$$2\sum_{k=1}^{(n-2)/2} r_k^2 + 4R^2 = \frac{n-2}{2}\left(4R^2\right) + 4R^2 = 2nR^2$$

which proves the theorem for every even n.This proof does not work if n is odd.

8. Proof for regular polygons with an odd number of sides

A different method that applies to all regular polygons with an odd number of sides is illustrated for a regular heptagon in Figure 9. The three segments and their mirror images in the diameter are the 6 segments drawn from one vertex of a regular heptagon to the other 6 vertices. We wish to prove that $2\left(r_1^2 + r_2^2 + r_3^2\right) = 14R^2$, or that

$$r_1^2 + r_2^2 + r_3^2 = 7R^2. \tag{7}$$

We apply the law of cosines to each of three isosceles triangles in Figure 9 having a vertex at the center of the heptagon, two edges of length R and base of length r_k, where $k = 1, 2, 3$. The corresponding vertex angles are θ_k, where $\theta_1 = 2\pi/7$, $\theta_2 = 4\pi/7$, $\theta_3 = 6\pi/7$. The law of cosines for the isosceles triangle with vertex angle θ_k states that

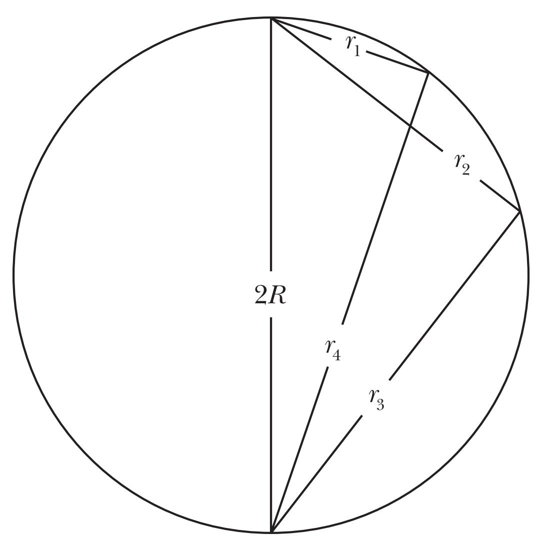

Figure 8. *Rearrangement of segments* r_1 *and* r_2 *in Figure 7.*

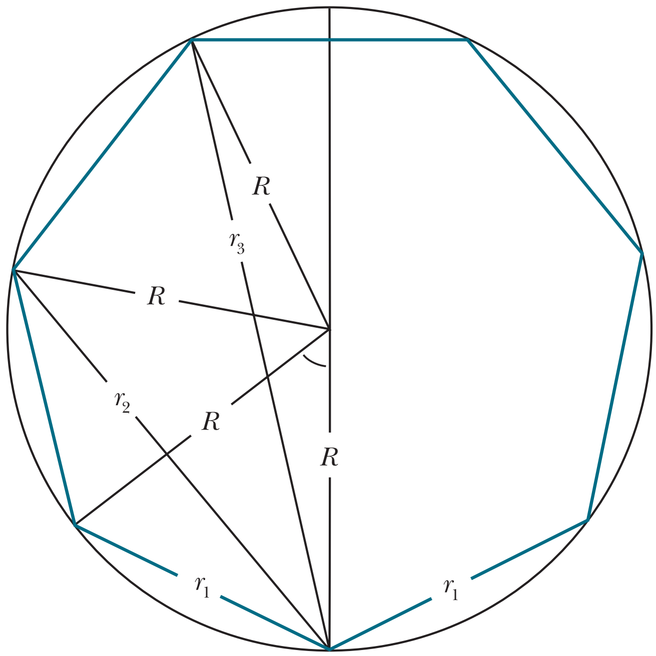

Figure 9. *Regular heptagon with edge r_1 inscribed in a circle of radius R.*

(8) $$r_k^2 = 2R^2 - 2R^2 \cos\theta_k$$

so the sum of these equations gives us

(9) $$\sum_{k=1}^{3} r_k^2 = 6R^2 - 2R^2 \sum_{k=1}^{3} \cos\theta_k\,.$$

But, by a trigonometric identity described below in (12), the sum of cosines is equal to –½, so (9) implies (7).

In the general case of a regular polygon with $2n+1$ sides we wish to prove that

(10) $$\sum_{k=1}^{n} r_k^2 = (2n+1)R^2.$$

In this case we apply the law of cosines to n isosceles triangles, using (8) for each of these triangles, where now $\theta_k = 2\pi k/(2n+1)$. Instead of (9) we have the equation

(11) $$\sum_{k=1}^{n} r_k^2 = 2(n-1)R^2 - 2R^2 \sum_{k=1}^{n} \cos\theta_k\,.$$

In this case we have the following identity (which we prove below in Section 9),

(12) $$\sum_{k=1}^{n} \cos\theta_k = -\frac{1}{2},$$

so (11) reduces to (10).

9. Origin of the trigonometric identity (12)

The trigonometric identity in (12) can be written as

(13) $$2\sum_{k=1}^{n} \cos\theta_k + 1 = 0$$

where $\theta_k = 2\pi k/(2n+1)$. This is a consequence of a more general trigonometric identity

(14) $$\sum_{k=1}^{m} \cos(2k\theta) = \frac{\sin m\theta \cos(m+1)\theta}{\sin\theta},$$

which holds for any positive integer m and any θ that is not an integer multiple of π. (See Exercise 32, p. 106, of Apostol's *Calculus,* Vol. I, 2nd ed., John Wiley & Sons, Inc, 1967.) If we take $\theta = \pi/m$ the right member vanishes and (14) becomes

(15) $$\sum_{k=1}^{m} \cos\left(\frac{2\pi k}{m}\right) = 0\,.$$

When m is odd, say $m = 2n + 1$ the last term in the sum is equal to 1. The remaining $2n$ terms can be arranged in n pairs, by coupling the terms with k and $m-k$, which have the same cosine. Consequently (15) can be written as

$$2\sum_{k=1}^{m} \cos\left(\frac{2\pi k}{2n+1}\right) + 1 = 0,$$

which is the same as (13).

Note. The foregoing method, using the law of cosines, also works if the polygon has an even number of sides, say $2n + 2$ sides, but one minor change is needed. There are now $2n + 1$ segments from a given vertex to the remaining vertices. One of these is a diameter, and the other $2n$ can be arranged in pairs by coupling each segment with its mirror image in that diameter. However, we do not give further details because the proof presented in Section 7 is more elementary.

10. Alternate proof for a regular pentagon

An alternate proof for a regular pentagon can be given by a method that is of interest because it makes use of Ptolemy's remarkable theorem on cyclic quadrilaterals (quadrilaterals inscribed in a circle). Ptolemy's theorem states that, for any cyclic quadrilateral, the product of the lengths of the diagonals is equal to the sum of the products of the lengths of opposite sides. (A simple proof of Ptolemy's theorem can be found in the Workbook that accompanies the videotape *Sines & Cosines, Part III*, produced by *Project MATHEMATICS!,* Caltech, 1994. The videotape also gives a computer animated version of this proof.)

Figure 10 shows a regular pentagon imbedded in a regular decagon with edges of length r_1. The segments r_2 and r_4 and their mirror images in a diameter $2R$ are the four segments drawn from one vertex of the regular pentagon to the other four vertices. We are to prove that $r_2^2 + r_4^2 + r_4^2 + r_2^2 = 10R^2$ or the equivalent statement

(16) $$r_2^2 + r_4^2 = 5R^2.$$

Apply Ptolemy's theorem to the cyclic quadrilateral in Figure 10 with two intersecting diagonals of length r_2 to obtain

(17) $$r_2^2 = r_1 r_3 + r_1^2\,.$$

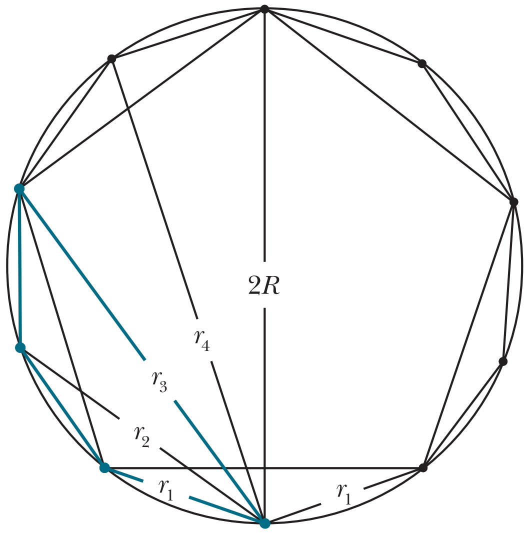

Figure 10. *A cyclic quadrilateral with three edges r_1, one edge r_3, and two diagonals r_2.*

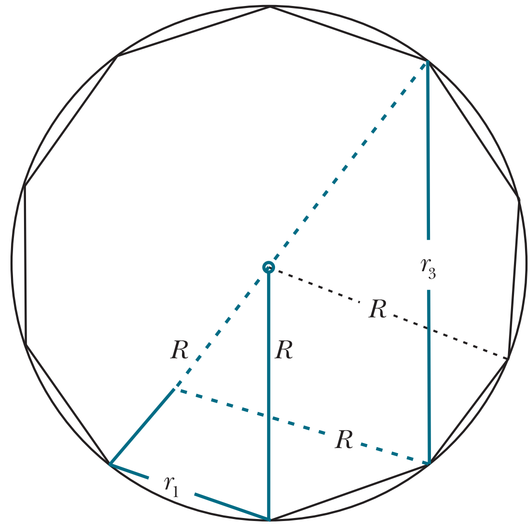

Figure 11. *The isosceles triangle with equal edges r_3 and base R is similar to that with equal edges R and base r_1.*

Next, refer to the two similar isosceles triangles shown in Figure 11, and equate ratios of corresponding sides to get $R/r_1 = r_3/R$, or

(18) $$r_1 r_3 = R^2.$$

Substitute (18) in (17) and then use the Pythagorean relation $r_1^2 + r_4^2 = (2R)^2$ to obtain

$$r_2^2 = R^2 + r_1^2 = R^2 + 4R^2 - r_4^2 = 5R^2 - r_4^2,$$

which implies (16). ■