Solids Circumscribing Spheres

Tom M. Apostol and Mamikon A. Mnatsakanian

1. INTRODUCTION. Although it is well known that every tetrahedron circumscribes a sphere, the following two simple consequences apparently have not been previously recorded. First, any plane through the center of the inscribed sphere divides the tetrahedron into two smaller solids whose surface areas are equal if and only if their volumes are equal. Second, the centroid of the boundary surface of a tetrahedron and the centroid of its volume are always collinear with the center of the inscribed sphere, at distances in the ratio 4:3 from the center.

This paper shows that both these and deeper results hold, not only for the tetrahedron or any polyhedron that circumscribes a sphere, but for more general solids called *circumsolids* (defined in section 4), whose faces can be curved as well as planar. The curved faces can be cylindrical, conical, or spherical. Each circumsolid circumscribes a sphere (its *insphere*), and all share the following property, proved in section 4:

Theorem 1. *The volume of any circumsolid is one-third the product of its outer surface area and the radius of its insphere.*

The discoveries in this paper are extensions to 3-space of corresponding planar results for *circumgons*—figures circumscribing circles—investigated in [**1**]. Section 2 reveals that Theorem 1 is implicitly contained in known formulas for volume and surface area of many familiar solids that happen to be circumsolids. Section 3 reviews the definition of circumgon, and section 4 extends it to 3-space. Section 5 applies Theorem 1 to interesting examples of circumsolids, including star-like circumsolids, and section 6 relates circumgons and circumsolids to isoperimetric problems. Section 7 uses Theorem 1 to solve the difficult problem of calculating the volume of the solid of intersection of a right circular cone cut orthogonally by a right circular cylinder. The result, stated in Theorem 6, appears to be new and generalizes the classical Archimedean result for the intersection of two circular cylinders. Section 8 relates the volume and surface area centroids of a circumsolid. Section 9 determines the volume of a circumsolid shell in terms of its constant thickness, thus providing a far reaching extension of the classical Egyptian and prismoidal formulas. Section 10 deals with centroids of circumsolid shells.

2. FAMILIAR CIRCUMSOLIDS. For any circumsolid, Theorem 1 tells us that $V/S = r/3$, where V is the volume, S is the outer surface area, and r is the radius of the insphere (called the *inradius*). The following examples illustrate this property with familiar solids that are also circumsolids.

Example 1 (Sphere). The known formulas $V = 4\pi r^3/3$ and $S = 4\pi r^2$ for a sphere of radius r reveal that $V/S = r/3$.

Example 2 (Prism circumscribing a sphere). Not every prism circumscribes a sphere. A notable exception is a right prism whose base is a regular n-gon. The base circumscribes a circle whose radius r is that of the inscribed sphere. If each edge of the base has length a, the lateral surface area is $2nar$, and the base has area $nar/2$, so the total surface area $S = 3nar$. Its volume V equals nar^2, which gives $V/S = r/3$.

Example 3 (Cylinder circumscribing a sphere). A right circular cylinder circumscribing a sphere of radius r has volume $V = 2\pi r^3$, lateral surface area $4\pi r^2$, and total surface area $S = 6\pi r^2$, implying that $V/S = r/3$. This also follows from Example 2, because the cylinder is a limiting case of a circumscribing prism.

Example 4 (Tetrahedron). Every tetrahedron, regular or not, circumscribes a sphere. It is known, and easy to verify, that its volume V and total surface area S are related by the formula $V = Sr/3$, where r is the inradius. In fact, if the four triangular faces have areas S_1, S_2, S_3, and S_4, then $S = S_1 + S_2 + S_3 + S_4$. On the other hand, the tetrahedron can be divided into four pyramids having a common vertex at the center of the inscribed sphere and respective volumes $V_k = S_k r/3$, whose sum is $V = Sr/3$, as asserted.

Theorem 1 yields a simple relation between the inradius and the four altitudes of any tetrahedron. Let h_k denote the altitude to the face of area S_k from the opposite vertex. Then $V = S_k h_k/3$, so $S_k = 3V/h_k$ for each k, and

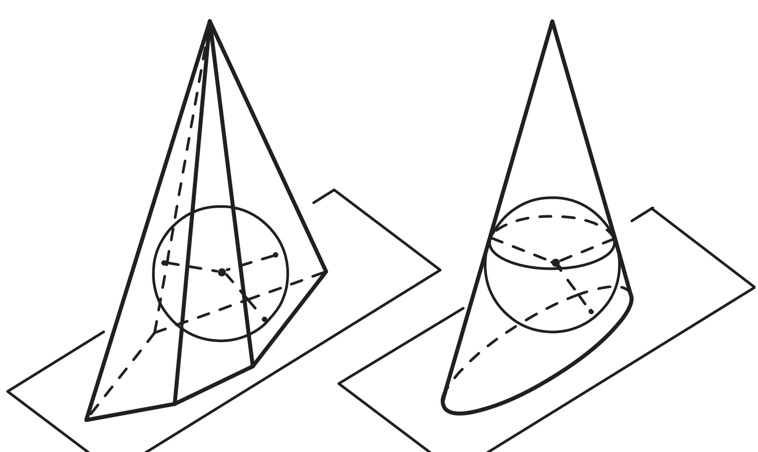
$$S = 3V\left(\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} + \frac{1}{h_4}\right).$$

But we also have $S = 3V/r$, hence

$$\frac{1}{r} = \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} + \frac{1}{h_4}.$$

In other words, *the reciprocal of the inradius of any tetrahedron is the sum of the reciprocals of its four altitudes*. This extends a corresponding result for the inradius of a triangle: *the reciprocal of the inradius of a triangle is the sum of the reciprocals of its three altitudes.*

Example 5 (Pyramid circumscribing a sphere). A right pyramid whose base is any regular polygon and whose altitude passes through the center of the base circumscribes a sphere. Its volume V and total surface area S are related by the formula $V = Sr/3$, where r is the inradius. This can be verified by dividing the pyramid into smaller pyramids with a common vertex at the incenter as was done in Example 4. In fact, the same method applies even if the polygonal base is not orthogonal to the axis of



(a) (b)

Figure 1. (a) Pyramid circumscribing sphere; (b) cone circumscribing sphere.

the pyramid, but slanted as shown in Figure 1a. Again, we find that $V = Sr/3$. For a right pyramid, this can also be derived (with more effort) from known formulas for the volume and surface area of a pyramid.

Example 6 (Cone circumscribing a sphere). As the number of edges of the polygonal base in Example 5 tends to ∞, the pyramid becomes a circular cone circumscribing the same sphere, as illustrated by Figure 1b. Consequently, the relation $V/S = r/3$ also holds for the limiting cone. For a right circular cone, this can also be derived from known formulas expressing volume and surface area in terms of the altitude and slant height of the cone, but additional effort is required to relate these quantities to the inradius r.

The volume and surface area of a right circular cone have been studied since antiquity. It seems astonishing that apparently no one has previously recorded the remarkable relation $V/S = r/3$ connecting its *total surface area*, volume, and inradius.

The foregoing examples illustrate the intrinsic presence of Theorem 1 in familiar circumsolids. To extend the applicability of Theorem 1 to less familiar circumsolids, we first recall the definition of a circumgon as given in [**1**].

3. BUILDING BLOCKS OF A CIRCUMGON. In [**1**], the definition of a general circumgonal region was formulated in terms of two simpler elements called *building blocks*, illustrated in Figure 2, and whose definition is repeated here. Start with a given circle (the incircle) and a triangular wedge with one vertex at the center (the incenter), whose side opposite the vertex is a segment on a line tangent to the circle. This wedge is called a *circumgonal building block*; the side opposite the center on the tangent line is called the *outer edge* of the block, as shown by the example in Figure 2a. The second type of building block, shown in Figure 2b, is any sector of the incircle itself (the limiting case of circumscribing polygons). Its outer edge is the circular arc of the incircle. The area of each building block, whether it is a triangular wedge or a circular sector, is half the length of its outer edge times the *inradius* (the radius of the incircle). We define the *perimeter* of the block to be the *length of its outer edge*; then the area of each building block is half its perimeter times its inradius.

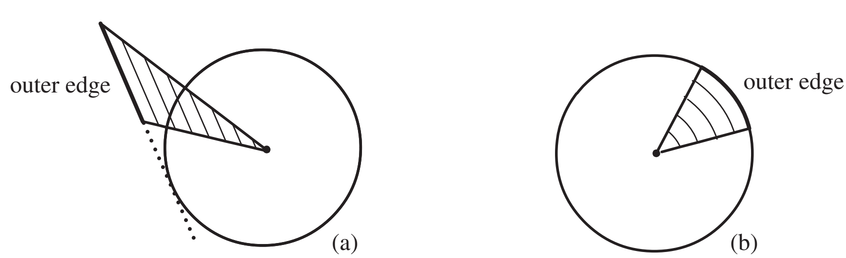


Figure 2. A building block of a circumgonal region is either (a) a triangular wedge or (b) a circular sector. Its perimeter is defined to be the length of the outer edge.

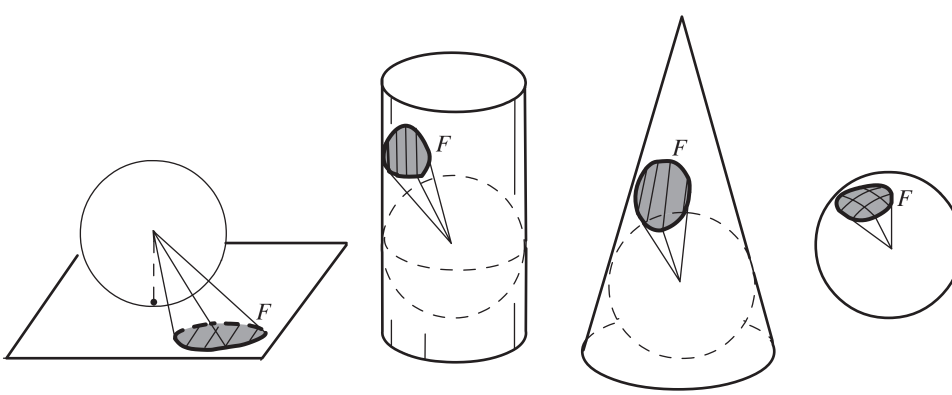
A general circumgonal region was defined in [**1**] as the union of a finite set of nonoverlapping building blocks having the same incircle. The union of the corresponding outer edges is called a *circumgon*; the sum of the lengths of the outer edges is defined as the *perimeter* of the circumgon. This immediately implies the planar version of Theorem 1:

The area of any circumgonal region is one-half its perimeter times its inradius.

4. BUILDING BLOCKS OF A CIRCUMSOLID. By analogy with circumgons, we define general circumsolids in terms of simpler building blocks. Instead of two types used in the planar case we consider four types, as illustrated in Figure 3. This leads to a class of circumsolids with more extensive applications than the class of circumgons treated in the planar case.

Flat-faced building block. Start with a given sphere (called the *insphere*), and a tangent plane. In this plane, consider a region F bounded by a simple closed curve, and assume that F has a finite area. Form the union of all line segments joining the incenter to the points of F. This is a conical solid having F as its base, the incenter as its vertex, and the inradius as its altitude. The term "conical solid" is used with the understanding that the solid is a pyramid when the base is polygonal. We call this solid a *flat-faced building block*, and we call F its *outer face* (Figure 3a). This is the 3-dimensional analog of the triangular wedge building block in Figure 2a, the outer face being the analog of the outer edge. We extend the analogy further by defining the outer surface area of the building block to be the area of its outer face. Because the volume of any conical solid is one-third the area of its base times its altitude, we see that the volume of any flat-faced building block is one-third the product of its outer surface area and its inradius.

Any *polyhedral* solid circumscribing a sphere is the union of a finite number of flat-faced building blocks, each of whose outer faces is polygonal. The surface area of a polyhedral solid is the sum of the areas of its polygonal faces, and its volume is the sum of the volumes of the blocks. Hence, Theorem 1 holds for every polyhedral circumsolid.



(a) flat-faced (b) cylindrical-faced (c) conical-faced (d) spherical-faced

Figure 3. Four types of building blocks for a circumsolid.

The most general circumsolid is one that can be obtained from flat-faced building blocks by a limiting process. Consider a region F of finite area lying on a surface all of whose tangent planes are also tangent to a sphere (the insphere), for example, on a developable surface with generators tangent to the insphere. Imagine the set of all line segments joining the incenter to points of F. This can be regarded as a building block with F as its *outer face*, and we call the area of F the *outer surface area* of the building block. A general surface tangent to a sphere is not easy to visualize. For simplicity, we consider only four familiar types of surfaces tangent to the insphere: a plane, a cylinder, a cone, and the sphere itself. These are especially suitable for applications. The plane gives flat-faced building blocks, and we turn now to the other three types.

Cylindrical-faced building block. In this type, shown in Figure 3b, the outer face F lies on a cylindrical surface tangent to the sphere. We call this a *cylindrical-faced building block*. Every such block is the limiting case of flat-faced building blocks, so its volume is one-third the product of its outer surface area and its inradius.

Conical-faced building block. For this type, shown in Figure 3c, region F lies on a conical surface tangent to the sphere. We refer to this type as a *conical-faced building block*. Every conical-faced building block is again the limiting case of flat-faced building blocks, ensuring that its volume is one-third the product of its outer surface area and its inradius.

Spherical-faced building block. This is the 3-dimensional analog of a circular sector building block that we label a *spherical-faced building block* (see Figure 3d). In this case, F lies on the surface of the insphere and plays the role of the circular arc intercepted by the circular sector in Figure 2b. As with the previous types, each spherical building block is the limiting case of flat-faced building blocks. Its volume is thus one-third the product of its outer surface area and its inradius.

Definition of circumsolid. A *circumsolid* is the union of a finite set of nonoverlapping building blocks having the same insphere. The sum of the areas of the outer faces is called the *outer surface area* of the circumsolid.

This definition leads to the following equivalent formulation of Theorem 1:

Theorem 1. *The ratio of volume to outer surface area of any circumsolid is one-third its inradius.*

In section 8 we shall use building blocks again to prove Theorem 7, which relates the volume centroid of any circumsolid to that of its boundary surface.

5. APPLICATIONS OF THEOREM 1. If a circumsolid has volume V and outer surface area S, Theorem 1 implies that $V = Sr/3$, where r is the inradius. The planar version for a circumgon of area A and perimeter P states that $A = Pr/2$, where r is the inradius. Both formulas have the same form:

$$V_n = \frac{r}{n} S_{n-1}. \tag{1}$$

For circumgons we have $n = 2, V_2 = A$, and $S_1 = P$. For circumsolids we have $n = 3$, $V_3 = V$, and $S_2 = S$. To avoid stating results separately for cirgumgons and circumsolids, we refer to both as n-circumsolids with volume V_n and outer surface area S_{n-1}, where n is either 2 or 3. The *incenter* is the center of the insphere. We use the term "insphere" with the understanding that it refers to the incircle if $n = 2$.

Many significant results flow from (1). For example, take any two n-circumsolids *with the same inradius* that have volumes V and V' and respective outer surface areas S and S'. From (1) we see that $V/V' = S/S'$, which immediately gives:

Theorem 2. *For any two n-circumsolids with the same inradius the ratio of their volumes is equal to the ratio of their outer surface areas.*

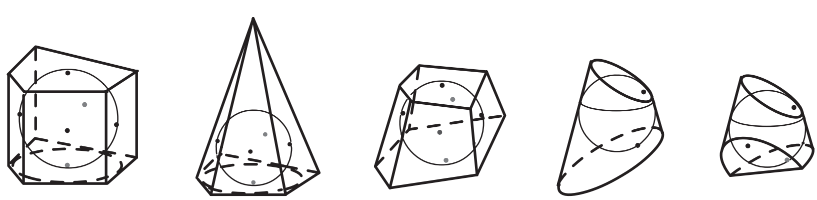
A striking consequence of Theorem 2 is obtained by cutting an n-circumsolid with an $(n-1)$-dimensional plane through its incenter. (We use the term "plane" rather than the more precise term "hyperplane," with the understanding that if $n = 3$ it means an ordinary plane, while if $n = 2$ it refers to a line in the plane of the circumgon.) Any finite set of planes passing through the incenter divides the solid into smaller circumsolids that have the same insphere but do not include the common $(n-1)$-dimensional faces in the dividing planes. Each pair of these circumsolids has the same inradius, so Theorem 2 has the following consequences:

Theorem 3. *A finite set of planes through the incenter of an n-circumsolid divides the circumsolid into smaller n-circumsolids in such a way that the ratio of the outer surface areas of any two is equal to the corresponding ratio of their volumes.*

Corollary. *A plane through the incenter of an arbitrary n-circumsolid bisects the outer surface area if and only if it bisects the volume. In this case, the two solids so formed (including the common face, if there is one) have equal total surface areas and equal volumes.*

For an arbitrary tetrahedron this gives the result mentioned in the opening paragraph of section 1, which we believe is new. The bisecting property for the case of a planar triangle is known [**3**]. Our simple proof, based on formula (1), makes the result seem almost trivial, not only for a triangle but for any circumgon.

The importance of Theorem 1 has been revealed by its connection with many familiar circumsolids and by its consequences in Theorems 2 and 3. But its real power comes into play when applied to a broader class of complicated circumsolids. For example, we can construct new families of circumsolids by replacing the regular polygonal bases of the prisms and pyramids in Examples 2 and 5 with circumgons, as suggested by the examples in Figures 4a and 4b. And we can form even more new families by truncating a given circumsolid by one or more planes tangent to the insphere, as indicated by the examples in Figures 4c, 4d, and 4e.



(a) (b) (c) (d) (e)

Figure 4. Further examples of circumsolids.

Our results can be used to analyze various intersections of circumsolids having the same insphere. For example, in section 7 we use Theorem 1 to calculate the volume of the solid of intersection of a right circular cone and an orthogonal cylinder with the same insphere.

Example 7 (Archimedean dome and circumscribing prism). Figure 5a shows an Archimedean dome, a circumsolid that is the union of cylindrical-faced building blocks. The cross section through the sphere's equator is a polygon circumscribing the equator, but the equatorial base is *not* one of the outer faces when the dome is regarded as a circumsolid. Each cross section by a plane parallel to the equator is a similar polygon circumscribing the circular cross section of the sphere and having the same orientation in space. Figure 5b shows the dome circumscribed by a prism, a circumsolid with the same insphere, whose cross sections are congruent to the equatorial polygon in Figure 5a.

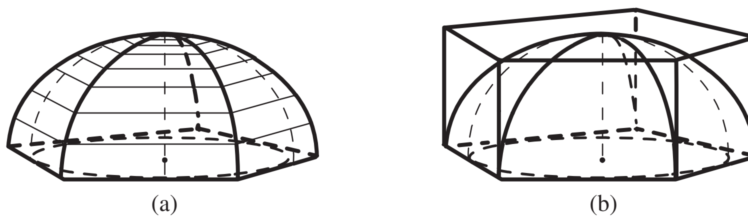


Figure 5. Archimedean dome (a) and its circumscribing prismatic container (b) (neither includes the base).

Archimedean domes and their circumscribing prismatic containers were introduced in [**2**], where it was shown that the ratio V_d/V_p of dome volume V_d to prism volume V_p is $2/3$ and that the same is true for the ratio S_d/S_p of lateral dome area S_d to outer prism area S_p (lateral area plus the area of the top face). This implies $V_d/S_d = V_p/S_p$, and by Theorem 1 this common ratio is $r/3$, where r is the inradius.

Example 8 (Star-shaped circumsolids). Like circumgons, circumsolids need not be convex and include star-shaped figures like stellated polyhedra obtained by extending the faces of regular convex polyhedra. Figure 6a shows a *stellated dodecahedron* formed by extending the edges of each pentagonal plane face of a regular dodecahedron until they form a *pentagram*. The stellated dodecahedron formed by the twelve intersecting pentagrams can also be constructed by adding twelve pyramids to the faces as indicated in Figure 6b. This is a circumsolid because each plane face is tangent to the insphere. More examples of star-shaped circumsolids can be formed by extending the outer plane faces of a convex polyhedral circumsolid or, for example, by suitably attaching cones to a given insphere. In each case, the ratio of volume to total surface area is one-third the inradius.

Both the stellated polyhedron and the core polyhedron circumscribe the same sphere. Hence, by Theorem 2, the ratio of their volumes is the same as the ratio of their total surface areas. This ratio, in turn, can be expressed in terms of planar ratios.

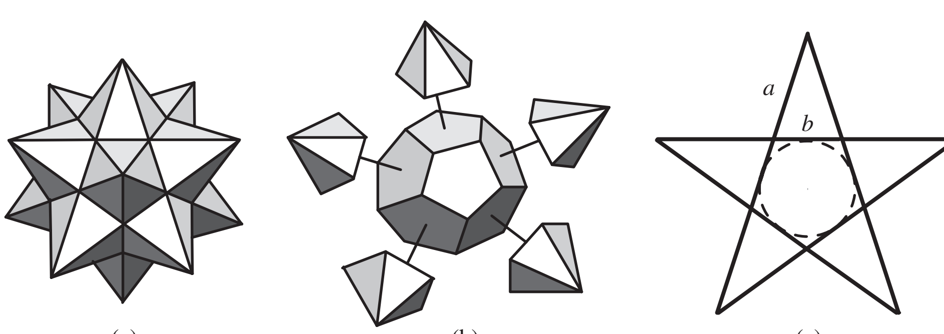


Figure 6. (a) Stellated dodecahedron; (b) pyramids added to a dodecahedron; (c) pentagram and pentagon.

To illustrate with an example, we refer to Figure 6c, which shows a pentagram with ten legs of length a circumscribing a pentagon of edge length b. If $\gamma = a/b$, the ratio of their perimeters is $10a/(5b) = 2\gamma$. Because the pentagram and pentagon circumscribe the same incircle, the ratio of their areas is also 2γ. The surface area of the stellated dodecahedron is equal to the total lateral surface area of the twelve pyramids, each of which consists of five congruent triangles with total area equal to that of the pentagram, minus the area of the pentagonal base. Consequently, the ratio of the surface area of the stellated dodecahedron to that of the dodecahedron is $2\gamma - 1$, which is also the ratio of their volumes. It is known that $\gamma = (1+\sqrt{5})/2$ (the golden ratio), so $2\gamma - 1 = \sqrt{5}$.

The simple calculations of Example 8 lead to the following observations:

$$\frac{\text{area of pentagram}}{\text{area of pentagon}} = \frac{\text{perimeter of pentagram}}{\text{perimeter of pentagon}} = 2\gamma = 1 + \sqrt{5},$$

$$\frac{\text{volume of stellated dodecahedron}}{\text{volume of dodecahedron}} = \frac{\text{surface area of stellated dodecahedron}}{\text{surface area of dodecahedron}}$$
$$= 2\gamma - 1 = \sqrt{5}.$$

Note that the use of Theorem 2 enabled us to deduce the volume and surface area relations by analyzing planar regions only.

6. OPTIMAL CIRCUMGONS AND CIRCUMSOLIDS. Two figures in the plane (or in 3-space) are said to have *the same shape* if one of them can be scaled to become congruent to the other. Thus, any two similar figures have the same shape. A typical isoperimetric problem in the plane compares two different shapes with equal perimeters and asks for the shape with larger area, which is called *optimal*. In 3-space, two solids with different shapes having a given surface area are compared, and the shape with larger volume is called optimal. This section treats optimal circumgons and circumsolids. The results are of particular interest because the figures need not be convex and our comparisons lead to quantitative relations as illustrated in Examples 9 and 10.

The basic formula (1) for any n-circumsolid is invariant under scaling, so the size of a figure plays no role in optimality considerations. Applying (1) to two n-circumsolids with volumes V and V', outer surface areas S and S', and inradii r and r', respectively, we find that

$$\frac{V}{V'} = \frac{S}{S'}\frac{r}{r'},$$

from which we deduce the following facts:

Theorem 4. *Suppose that two different n-circumsolids have equal outer surface areas. If the ratio of their inradii is ρ, then the ratio of their volumes is also ρ. The one with the larger inradius has larger volume, hence is optimal.*

Corollary. *Among all n-circumsolids with a given outer surface area, the n-sphere has the largest inradius and the largest volume, hence is optimal.*

Theorem 4 not only establishes optimality, which is a qualitative result, but it also gives a quantitative comparison of areas and volumes. This is demonstrated by the next two examples.

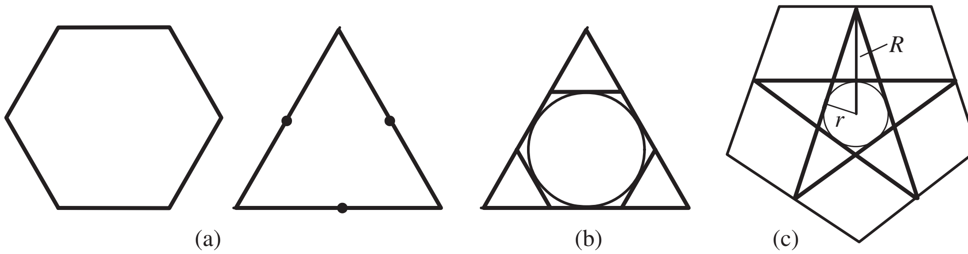


Figure 7. (a) Isoperimetric hexagon and triangle; (b) hexagon inscribed in triangle; (c) Example 10.

Example 9 (Equilateral triangle and hexagon). Figure 7a shows a regular hexagon and an equilateral triangle having equal perimeters. Because the hexagon is more "circular" than the triangle, it has larger area. What is the exact ratio of their areas? The two-dimensional version of Theorem 4 answers this question. The ratio of their areas is equal to the ratio of their inradii, so we need to determine the ratio of the inradii. One way to do this is illustrated in Figure 7b, which shows a smaller regular hexagon inscribed in the equilateral triangle. Both are circumgons with the same incircle, and it is clear from Figure 7b that the ratio of their perimeters (triangle to hexagon) is 9 to 6, or $3/2$. Therefore if we scale the inscribed hexagon from its incenter by the factor $3/2$, we obtain the larger hexagon in Figure 7a having the same perimeter as the triangle. The larger hexagon and the original triangle are circumgons whose inradii have ratio $3/2$, so by Theorem 4 the ratio of their areas is also $3/2$. In other words, *a regular hexagon with the same perimeter as an equilateral triangle has three-halves the triangle's area*. This can also be seen directly by dissecting each polygon in Figure 7a into smaller equilateral triangles.

Example 10 (Pentagram and regular pentagon). Suppose that a pentagram and a regular pentagon have equal perimeters, as in Figure 7c. What is the exact ratio of their areas? The midpoints of the edges of the large regular pentagon are the vertices of the pentagram, and it is clear that the large pentagon does have the same perimeter P as the pentagram. The large pentagon is a circumgon with inradius R and area $PR/2$, and the pentragram is a circumgon with inradius r and area $Pr/2$, so the ratio of their areas is R/r, the ratio of their inradii, as predicted by Theorem 4. From similar triangles in Figure 7c we obtain $R/r = a/(b/2) = 2\gamma$ (see Example 8), yielding an area ratio of 2γ:

$$\frac{\text{area of large pentagon}}{\text{area of pentagram}} = 2\gamma = 1 + \sqrt{5}.$$

The same result can be obtained without explicitly constructing the large pentagon. The pentagram and the smaller pentagon have the same inradius, and we found in Example 8 that the ratio of their perimeters is 2γ. Therefore, if we expand the small pentagon from the incenter by the scaling factor 2γ, we obtain a larger pentagon with the same perimeter as the pentagram. The ratio of their inradii is 2γ, hence by Theorem 4 the ratio of their areas is also 2γ. The use of Theorem 4 is preferable because it applies in cases in which it is not clear how to construct explicitly a polygon isoperimetric to another.

The next theorem refers to two n-circumsolids with the same inradius. Theorem 2 tells us that the ratio of their volumes is equal to the ratio of their surface areas. Let μ denote this common ratio (larger to smaller), so that $\mu \geq 1$.

Theorem 5. *If two different n-circumsolids have the same inradii, then the one with smaller volume (or smaller outer area) has optimal shape. Moreover, for the same outer area the optimal shape has volume exactly μ times larger than the other shape.*

Proof. Expand the smaller solid from its incenter by the scaling factor μ to match the outer area of the larger solid. Expansion increases the inradius by the factor μ. In view of Theorem 4, its volume is μ times larger, and its shape is optimal. ■

Example 11 (Stellated dodecahedron). In Example 8 we showed that the volume of the stellated dodecahedron is $\sqrt{5}$ times that of the dodecahedron with the same insphere. Therefore, according to Theorem 5 with $\mu = \sqrt{5}$, for the same outer surface area a dodecahedron has $\sqrt{5}$ times larger volume than a stellated dodecahedron.

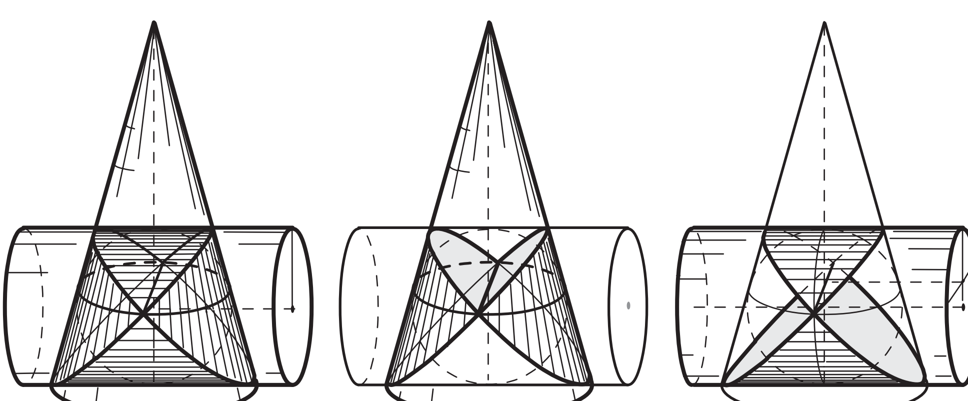
Similarly, the results of Examples 9 and 10 follow immediately from Theorem 5.

7. INTERSECTION OF A CONE AND CYLINDER WITH THE SAME INSPHERE. Figure 8a shows a circular cone with a vertical axis and a circular cylinder with a horizontal axis circumscribing the same insphere of radius R.

Problem. *Find the volume of the solid of intersection of this cone and cylinder.*

The corresponding problem for two intersecting cylinders originated with Archimedes and has become a standard calculus exercise that is relatively easy to solve because all cross sections in one direction are squares. The result (see [**2**, Theorem 4b]) is two-thirds the volume of its circumscribing cube, or $16R^3/3$.

Calculating the volume of the solid of intersection of a cone and cylinder is much more difficult. In Figure 8a, each horizontal cross section perpendicular to the axis of the cone is a rectangle capped by circular segments on two opposite edges. This cross-sectional area can be expressed as a complicated function of the vertical distance from the center of the insphere and the volume V of the solid of intersection can be expressed as an integral of the cross-sectional areas. This approach leads to some unattractive integrals that are not easy to evaluate. They can be completely avoided by using Theorem 1.



(a) (b) (c)

Figure 8. (a) Solid of intersection; (b) bounded by a conical surface; (c) a cylindrical surface.

The solid in question is a circumsolid, and Theorem 1 tells us that $V = RS/3$, where S denotes the area of its outer boundary surface. This reduces the problem to that of calculating S.

The boundary surface in Figure 8a consists of two parts, a conical portion (Figure 8b) whose area we call S_1 and a cylindrical portion (Figure 8c) whose area we call S_2, each of which can be calculated separately without integration. We describe these calculations in detail because they involve geometric properties of the boundary surface that are of independent interest. The final results are stated as follows:

Theorem 6. *The intersection of a right circular cone with vertex angle 2α and an orthogonal circular cylinder circumscribing the same insphere of radius R is a circumsolid. Its boundary surface has a conical portion of area S_1 given by*

$$S_1 = 4R^2\left(1 + \frac{2\alpha}{\sin 2\alpha}\right) \tag{2}$$

and a cylindrical portion of area S_2 given by

$$S_2 = 4R^2(2 + 2\alpha\tan\alpha). \tag{3}$$

The volume of the solid of intersection is $V = R(S_1 + S_2)/3$ or

$$V = \frac{4}{3}R^3\left(3 + 2\alpha\tan\alpha + \frac{2\alpha}{\sin 2\alpha}\right). \tag{4}$$

Remark. Keep R fixed in (4) and let $\alpha \to 0$. The cone becomes a cylinder of radius R, and the solid becomes the intersection of two orthogonal cylinders (an Archimedean globe) with volume $16R^3/3$, as expected.

The proof of Theorem 6 depends on a sequence of lemmas. The first, whose proof is left to the reader, deals with known properties of an ellipse and does not involve parameters associated with the cone.

Lemma 1. (a) *An ellipse with semiaxes of lengths A and B (where $B \le A$) has eccentricity $\sqrt{1-(B/A)^2}$.*

(b) *If the ellipse lies on a plane inclined at angle β from the horizontal, then its projection onto a vertical plane through its minor axis is another ellipse with semiaxes of lengths B and $A\sin\beta$.*

(c) *The projected ellipse in* (b) *is a circle if and only if $B/A = \sin\beta$, in which case the inclined ellipse has eccentricity* $\cos\beta$.

The next lemma refers to an ellipse of eccentricity $\cos\beta$ cut from a right circular cone with vertex angle 2α by a plane inclined at angle β from its horizontal base, as shown in Figure 8. Because the eccentricity is $\cos\beta$, its vertical projection is a circle, and the circular cylinder having this circle as profile intersects the cone along this ellipse and along a symmetric congruent ellipse, as depicted in Figure 8. The cone and cylinder are circumsolids with a common insphere whose radius we denote by R.

Figure 9a shows a vertical cross section of this cone and cylinder, cut by a plane through the axis of the cone. Point O is the incenter, and point C is the center of one of the two slanted ellipses with major axis QP. (The corresponding diagram for the congruent slanted ellipse is not shown.) The length of segment CO is denoted by c, written as follows: $|CO| = c$. When the cone and cylinder circumscribe the same

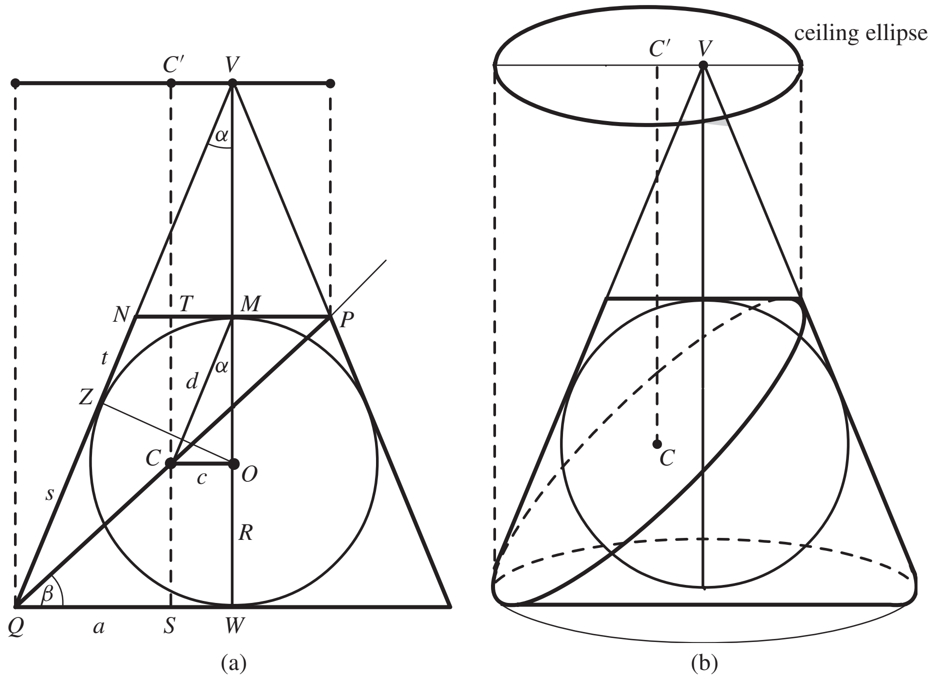


Figure 9. (a) Vertical cross section of the intersection of cone and cylinder circumscribing the same insphere; (b) the ceiling ellipse is the vertical projection of the slanted ellipse onto a horizontal plane.

insphere, angles α and β are related in a manner described by the next lemma, which also expresses various distances on the ellipse in terms of the inradius R.

Lemma 2. (a) *Angles α and β satisfy the relation*

$$\cos\alpha = \tan\beta. \tag{5}$$

(b) *The distance $c = |CO|$ is given by*

$$c = R\tan\alpha. \tag{6}$$

(c) *The semiaxes of the slanted ellipse have lengths R and $R/\sin\beta$.*

Proof. First we introduce some notation. In Figure 9a, M is the midpoint of NP, and C is the midpoint of QP. Therefore MC is parallel to NQ, hence angle OMC is equal to α. A vertical line through O intersects the base at W, and a vertical line through C intersects NP at T and QW at S. Point Z is where the incircle touches NQ. Therefore segments NM and NZ have equal lengths, which we denote by t, while segments QZ and QW have equal lengths that we denote by s. Finally, let $a = |QS| = |TP|$, and let $d = |MC|$. Now $c = |CO| = |TM|$, implying that $t = a - c$. Because MC is parallel to NQ, $2d = s + t$. But $s = a + c$ and $t = a - c$, so $2d = s + t = (a + c) + (a - c) = 2a$, whence $a = d$.

Next, the right triangle COM shows that $R = d\cos\alpha$, and the right triangle QSC shows that $R = a\tan\beta$. Because $a = d$, we get (5). Triangle COM also gives (6).

The length of the semiminor axis of the ellipse is R. A glance at the triangle QSC reveals that the semimajor axis has length $A = |QC| = R/\sin\beta$. ∎

Ceiling ellipse. The projection of the slanted elliptical cross section of the cone onto the ceiling plane, a horizontal plane through vertex V, is called a *ceiling ellipse* (Figure 9b). Its semiminor axis has length R; its semimajor axis has length $a = A\cos\beta = R/\tan\beta$. The center O of the incircle projects onto the vertex V, and the center C of the ellipse projects onto a point we denote by C'.

Lemma 3. *The ceiling ellipse has eccentricity* $\sin\alpha$ *and has one of its foci at* V.

Proof. First we determine the eccentricity. The ceiling ellipse has semiminor axis of length R and semimajor axis of length $R/\tan\beta$, so by Lemma 1a its eccentricity is $\sqrt{1-\tan^2\beta}$. In light of (5), this is $\sqrt{1-\cos^2\alpha} = \sin\alpha$.

To prove that V is a focus, it suffices to prove that

$$|C'V| = \sin\alpha(R/\tan\beta), \tag{7}$$

which says that $|C'V|$ is the product of the eccentricity times the length of the semimajor axis. We infer from (6) that $|C'V| = c = R\tan\alpha = R\sin\alpha/\cos\alpha$. By (5) the latter expression is equal to $R\sin\alpha/\tan\beta$, and we obtain (7). ∎

Lemma 4. *Suppose that a right pyramid has a regular n-gon as base and altitude through the incenter of the base. If* α *signifies the angle each face makes with the altitude, then any region of area S on the lateral surface projects onto a plane region of area* $S\sin\alpha$ *on the ceiling plane. The same is true of any region of area S on the lateral surface of a right circular cone with vertex angle* 2α.

Proof. Each triangular face of the lateral surface of the pyramid with area T projects onto a triangle of area $T\sin\alpha$ in the ceiling plane. Hence any subregion of the lateral surface of the pyramid with area S has a ceiling projection of area $S\sin\alpha$. This property is independent of n, and we obtain the result for a cone by letting $n \to \infty$. ∎

Calculation of the conical portion S_1. We refer to Figure 10a. In the ceiling plane the shaded region is the ceiling projection of the conical portion of area S_1 and has area equal to $S_1\sin\alpha$. This area, in turn, is the sum of the areas of two confocal overlapping ellipses, minus twice the area of their intersection. In other words, $S_1\sin\alpha =$

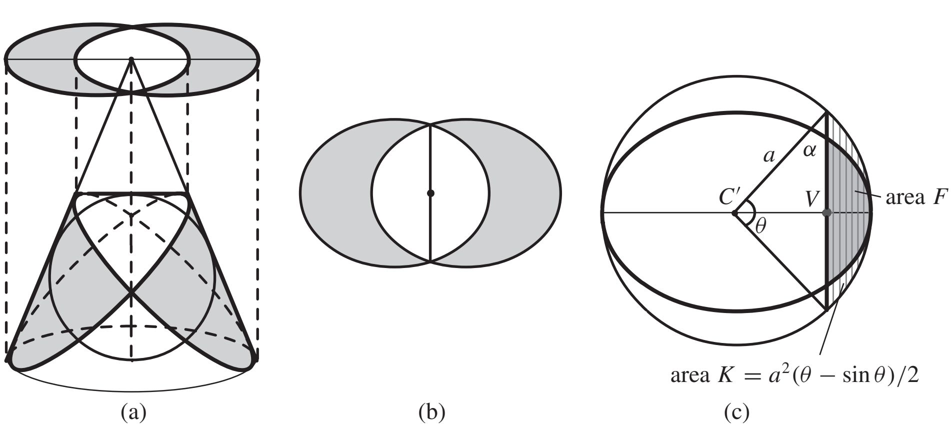


Figure 10. (a) Two conical wedges cut by inclined planes; (b) ceiling projection; (c) calculation of F.

$2(E - I)$, where E is the area of the region bounded by each ceiling ellipse and I is the area of the intersection (Figure 10b). Area I, in turn, is equal to $2F$, where F is the area of the focal segment cut from the ellipse by a chord through its focus perpendicular to the major axis. Hence we have

$$S_1 \sin\alpha = 2E - 4F. \tag{8}$$

Now we calculate E and F separately. Each ceiling ellipse has semiaxes of lengths R and $R/\tan\beta$, so $E = \pi R^2/\tan\beta$, and from (5) we find that

$$E = \frac{\pi R^2}{\cos\alpha}. \tag{9}$$

The next lemma evaluates F in terms of R and α.

Lemma 5. *The elliptical focal segment has area F given by*

$$F = R^2\left(\frac{\pi - 2\alpha}{2\cos\alpha} - \sin\alpha\right). \tag{10}$$

Proof. In Figure 10c the focal segment of area F is shown shaded. Inscribe the ceiling ellipse in a circle with center at C' and with radius equal to a, the length of the semi-major axis. Because the ellipse has eccentricity $\sin\alpha$, a radial line from C' intersects the vertical chord through the focus V at an angle α. This radial line and its mirror image subtend a central angle $\theta = \pi - 2\alpha$, and the circular segment cut by the vertical chord through V has area $K = a^2(\theta - \sin\theta)/2$. But $a = R/\cos\alpha$, which leads to

$$K = \frac{1}{2}\left(\frac{R}{\cos\alpha}\right)^2(\theta - \sin\theta) = R^2\left(\frac{\pi - 2\alpha}{2\cos^2\alpha} - \frac{\sin 2\alpha}{2\cos^2\alpha}\right) = R^2\left(\frac{\pi - 2\alpha}{2\cos^2\alpha} - \frac{\sin\alpha}{\cos\alpha}\right).$$

Now $\cos\alpha$ is the dilation factor in the vertical direction that converts the circular segment to the elliptical focal segment, so $K\cos\alpha = F$, and we obtain (10). ∎

Finally, use (10) and (9) in (8), then divide by $\sin\alpha$, to obtain (2) in Theorem 6.

Calculation of the cylindrical portion S_2. First we recall a known property of an ellipse unwrapped from a right circular cylinder, as discussed in [**2**, sec. 8]. Cut a right circular cylinder of radius R by a plane through a diameter of its base at an angle of inclination γ, where $0 < \gamma < \pi/2$. The example in Figure 11a shows part of the elliptical cross section and a wedge cut from the cylinder. A vertical cutting plane parallel to the major axis of the ellipse intersects the wedge along a right triangle T (shown shaded), with base angle γ. When the surface of the cylinder is unwrapped onto

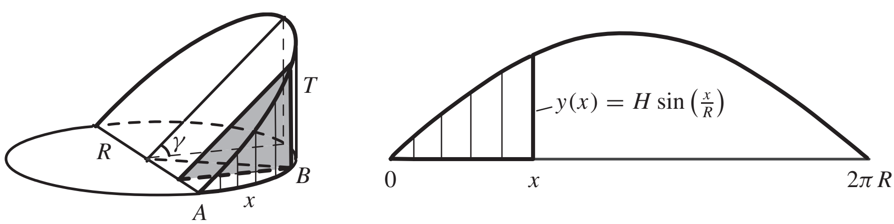


Figure 11. The circular arc AB of length x in (a) unwraps onto the line segment $[0, x]$ in (b); the altitude of triangle T unwraps onto the height $y(x) = H\sin(x/R)$.

a plane, the circular base unfolds along a line we call the x-axis. Here x is the length of the circular arc measured from point A at the extremity of the base diameter to point B at the base of triangle T, as shown in Figure 11a. The base of T has length $R\sin(x/R)$, and its height is $H\sin(x/R)$, where $H = R\tan\gamma$. Therefore the unwrapped curve is the graph of the sinusoidal function y with period $2\pi R$ and amplitude H given by $y(x) = H\sin(x/R)$.

To adapt the same idea to the solid in Figure 8, tip the solid so that the cylinder's axis is vertical, as in Figure 12. The cutting plane is inclined at an angle $\gamma = \pi/2 - \beta$ with a horizontal circular cross section of the cylinder and passes through a chord of the circle at vertical distance h, say, from the diameter. It cuts the lateral surface of the cylinder into two portions, the sum of whose areas is half the area S_2 in question. When unwrapped onto a plane, they form two shaded regions bounded by the curve $y = H\sin(x/R)$ and the horizontal line $y = h$. The area of the shaded region below the sine curve and above the line $y = h$ is denoted by A, while that of the region below the line $y = h$ and above the sine curve is denoted by B. The lateral surface area S_2 is $2(A + B)$.

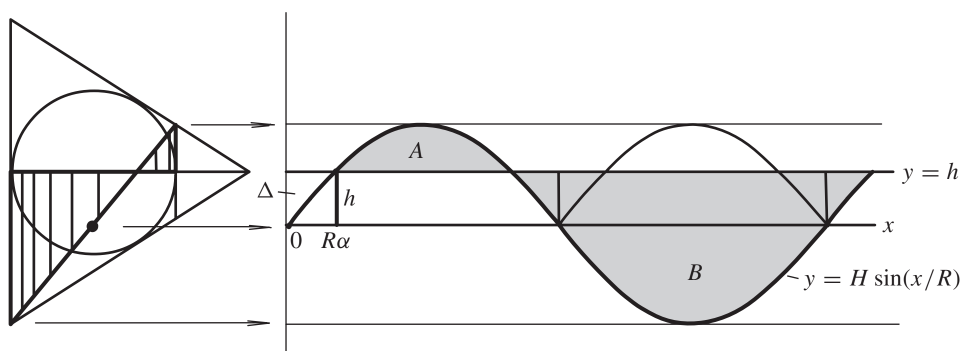


Figure 12. Lateral surface area S_2 is twice the sum $A + B$ of the areas of the shaded sinusoidal regions.

Twice the area of the region under one arch of the sine curve $y = H\sin(x/R)$ is $4HR$, and Figure 12 shows that $4HR = A + B - 4\Delta$, where Δ is the area of the curvilinear triangular region above the portion of the sine curve over the interval $[0, R\alpha]$ and below the line $y = h$. Distance $R\alpha$ is the length of the unwrapped circular arc of radius R subtending angle α (see Figure 13).

Area Δ, in turn, is the area of a rectangle of base $R\alpha$ and altitude h, minus the area of the curvilinear triangular region below the sine curve. Because $h = H\sin\alpha$ we discover (by [**1**, Corollary 12]) that

$$\Delta = R\alpha h - HR(1 - \cos\alpha),$$

hence

$$A + B = 4HR + 4\Delta = 4RH(\alpha\sin\alpha + \cos\alpha).$$

Now $H = R\tan\gamma = R\tan(\pi/2 - \beta) = R\cot\beta = R/\cos\alpha$ as a result of (5). Therefore,

$$A + B = 4R^2(\alpha\tan\alpha + 1),$$

which, when doubled, gives (3) and completes the proof of Theorem 6.

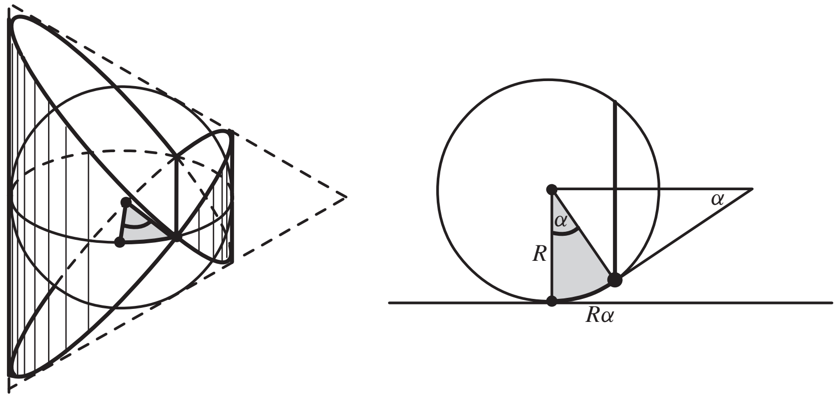


Figure 13. The unwrapped circular arc of radius R subtends an angle α, so the arclength is $R\alpha$.

8. CENTROIDS OF CIRCUMSOLIDS. In [**1**] we discovered a simple but surprising connection between the area centroid of a circumgon and the centroid of its boundary: *they are collinear with the incenter, at distances in the ratio* $3:2$ *from the incenter*. Now we deduce a corresponding result for circumsolids. Specifically, denote by $\boldsymbol{C}(S)$ the vector from the incenter O to the centroid of its outer boundary surface and by $\boldsymbol{C}(V)$ the vector from O to the volume centroid of the solid. For a given circumsolid the location of one of the centroids determines the location of the other. In fact, we have:

Theorem 7. *The volume centroid* $\boldsymbol{C}(V)$ *of any circumsolid and the centroid* $\boldsymbol{C}(S)$ *of its outer boundary surface are collinear with the incenter and are related by the equation*

$$\boldsymbol{C}(S) = \frac{4}{3}\boldsymbol{C}(V). \tag{11}$$

Proof. Consider first a flat-faced pyramidal building block whose outer face is a triangle. The centroid $\boldsymbol{C}(S)$ of the outer triangular face is in the plane of the triangle (at the intersection of the three medians). The centroid $\boldsymbol{C}(V)$ of the pyramid with this outer face as base has its centroid at a distance three-fourths the distance from the vertex to the centroid of the base. Because $\boldsymbol{C}(V)$ lies on the line segment joining the incenter to $\boldsymbol{C}(S)$ we have $\boldsymbol{C}(V) = (3/4)\boldsymbol{C}(S)$, which implies (11) for each pyramidal building block with a triangular outer face.

Now take any flat-faced circumsolid with polygonal outer faces and divide each outer face into triangular regions having outer areas $S_1, \dots, S_n$, a common vertex at the incenter O, and respective volumes $V_1, \dots, V_n$. Denote by $\boldsymbol{C}(V_1), \dots, \boldsymbol{C}(V_n)$ the corresponding vectors from the incenter O to the volume centroid of each pyramidal block with a triangular outer face. The vectors from O to the volume centroid $\boldsymbol{C}(V)$ of their union and to the area centroid $\boldsymbol{C}(S)$ of the boundary are given by

$$\boldsymbol{C}(V) = \frac{\sum_{k=1}^{n} V_k \boldsymbol{C}(V_k)}{\sum_{k=1}^{n} V_k}, \qquad \boldsymbol{C}(S) = \frac{\sum_{k=1}^{n} S_k \boldsymbol{C}(S_k)}{\sum_{k=1}^{n} S_k}. \tag{12}$$

In the first fraction, use $V_k = S_k r/3$, where r is the inradius, and in the second fraction apply (11) to each pyramidal block to find $\boldsymbol{C}(S_k) = (4/3)\boldsymbol{C}(V_k)$. Then (12) implies (11) for a polyhedral circumsolid. Because the other types of building blocks can be regarded as limiting cases of polyhedral circumsolids, formula (11) also holds for all four types of building blocks and hence for all circumsolids. ∎

Theorem 7 was stated for a tetrahedron in the opening paragraph of section 1. It extends Theorem 6 of [**1**], a corresponding result for circumgons that contains the fraction $3/2$ in place of $4/3$.

Example 12 (Archimedean dome and circumscribing prism). Recall the Archimedean dome and its circumscribing prism shown in Figure 5. In [**2**, Corollary 15] it was shown that the centroid of the surface of an Archimedean dome is at the midpoint of its altitude. If the dome is regarded as a circumsolid with inradius r, its outer surface area centroid is at distance $r/2$ from the base. Therefore, by Theorem 7, the volume centroid of the dome is at distance $3r/8$ from the base. This generalizes a result found by Archimedes for a hemisphere. Actually, the result for the volume centroid also holds for each component wedge of the Archimedean dome. Moreover, if each wedge of inradius r is dilated vertically by a factor λ to form a semielliptical wedge of altitude $h = \lambda r$, the elongated wedge is no longer a circumsolid, but its volume centroid is scaled by the same factor λ to a distance $3h/8$ from the horizontal base.

The circumscribing prism in Figure 5b has its volume centroid at distance $r/2$ from the base. Hence, by Theorem 7, the outer surface area of this circumsolid (lateral surface area plus the top face) has its centroid at a distance $2r/3$ from the base.

Example 13 (Right circular cone). The volume centroid of a right circular cone of altitude h is known to be on its axis at a distance $h/4$ from the cone's base. If the inradius is r, the volume centroid is a distance $r - h/4$ from the incenter. According to Theorem 7, the area centroid of the total surface of the cone is at a distance $4(r - h/4)/3$ from the incenter, hence at height $(h - r)/3$ from the base of the cone. In other words, the height of the centroid of the total surface area of a cone above its base equals one-third the distance between the incenter and the vertex. The same is true for any right pyramid with a circumgonal base and altitude passing through the incenter of the base.

9. CIRCUMSOLID SHELLS. Circumsolid shells are analogous to the circumgonal rings in the plane introduced in [**1**]. For any solid Q and any scalar λ with $0 < \lambda < 1$ choose a point O in Q, let λQ denote the solid consisting of all points $\lambda \boldsymbol{q}$ scaled from O as $\boldsymbol{q}$ ranges through all points of Q, and let Q_λ denote the solid shell lying between λQ and Q. The prototype is a spherical shell between two concentric spheres of radii r and λr.

We are interested in the case in which Q is a circumsolid with incenter O and inradius r. Then λQ is also a circumsolid with the same incenter and with inradius λr. The inner and outer boundary surfaces of a circumsolid shell Q_λ are "parallel," in the sense that the perpendicular distance between them is a constant, equal to $(1 - \lambda)r$, where r is the inradius of the larger circumsolid Q. This constant, which we denote by w, is called the thickness of the shell:

$$w = (1 - \lambda)r. \tag{13}$$

This proves part (a) of the following theorem:

Theorem 8. (a) *Every circumsolid shell has constant thickness.*

(b) *Conversely, any solid shell with constant thickness lying between two similar solids λQ and Q is necessarily a circumsolid shell.*

Theorem 8 extends Theorem 4 of [**1**] for circumgonal rings. Part (b) can be proved by passing a plane through the center of similarity perpendicular to each pair of faces, thus reducing the problem to the 2-dimensional case. We omit the details.

If the outer solid Q of a shell has area S and volume V, the inner solid λQ has area $T = \lambda^2 S$ and volume $\lambda^3 V$. The shell Q_λ itself has total surface area

$$S' = S + T = (1 + \lambda^2)S,$$

and volume V', where

$$V' = (1 - \lambda^3)V = (1 - \lambda)(1 + \lambda + \lambda^2)V.$$

If the outer solid is a circumsolid with volume V, then $V = Sr/3$, from which we find using (13) that

$$V' = \frac{r}{3}(1 - \lambda)(1 + \lambda + \lambda^2)S = \frac{w}{3}(S + \lambda S + \lambda^2 S).$$

But $\lambda^2 S = T$, so $\lambda = \sqrt{T/S}$, and the formula for V' becomes

$$V' = \frac{w}{3}(S + \sqrt{ST} + T). \tag{14}$$

This has the same form as the famous *Egyptian formula* for the volume of a truncated square pyramid, where S and T are areas of the square planar bases. In our version of (14), the inner and outer faces need not be planar. It can be used, for example, to calculate the volume of a hemisphere, taken as a circumsolid shell with inner surface area $T = 0$.

In (14), the term $\sqrt{ST}$, the geometric mean of S and T, is called the *mixed area* of S and T. The quantity $(S + \sqrt{ST} + T)/3$ that multiplies w is the average of S, T, and the mixed area $\sqrt{ST}$. We give it its own name:

Definition. We call the quantity $(S + \sqrt{ST} + T)/3$ the *mixed average surface area* of the circumsolid shell, and denote it by S_{ave}.

Thus, (14) reduces to $V' = wS_{\text{ave}}$. In other words, we have:

Theorem 9. *The volume of any circumsolid shell is the product of its mixed average surface area and its thickness.*

Theorem 9 extends Theorem 5 of [**1**], which states the area of any circumgonal ring is one-half the product of its total perimeter and its width.

It is not easy to interpret the mixed area term $\sqrt{ST}$ in (14) geometrically in terms of areas. But (14) can be written as equation (15), in which all terms refer to areas. This alternative form involves the area of the surface *midway* between the outer and inner surfaces, that is, the surface whose inradius is $r(1 + \lambda)/2$, the average of λr and r. We call the area of this midway surface the *midway area* and denote it by $S_{1/2}$. Then $S_{1/2} = (1 + \lambda)^2 S/4$, so $4S_{1/2} = (1 + \lambda)^2 S = S + 2\lambda S + \lambda^2 S = S + 2\sqrt{ST} + T$, from which we infer that

$$4S_{1/2} + S + T = 2(S + \sqrt{ST} + T) = 6S_{\text{ave}}.$$

We now use (14) to obtain the following extension of the classical *prismoidal formula*:

Theorem 10. *A circumsolid shell of thickness w, outer area S, inner area T, and midway area $S_{1/2}$ has volume V' given by*

$$V' = \frac{w}{6}(S + 4S_{1/2} + T). \tag{15}$$

In (15), the term multiplying w is a weighted arithmetic mean of areas S, T, and $S_{1/2}$. When the circumsolid shell is flat-faced with parallel outer planar faces at distance w apart, (15) becomes the classical prismoidal formula.

Because of Theorem 8b, circumsolid shells are the most general shells with curved surfaces for which *both* the Egyptian formula and the prismoidal formula give the exact volume. For example, the volume of a hemisphere satisfies both (14) and (15) when it is considered as a circumsolid shell (with inner surface area $T = 0$). By contrast, with standard use of *planar* cross sections, the prismoidal formula gives the correct answer, but the Egyptian formula does not. Another example is a general solid angle or a truncated version thereof.

10. CENTROIDS OF CIRCUMSOLID SHELLS. A companion result to Theorem 7 relates the volume centroid of the circumsolid shell Q_λ to that of the outer circumsolid Q. It extends Theorem 8 of [**1**], a corresponding result for circumgonal rings.

Theorem 11. *The volume centroid $\boldsymbol{C}(Q_\lambda)$ of the circumsolid shell Q_λ is related to the volume centroid $\boldsymbol{C}(Q)$ of the outer circumsolid Q by the equation*

$$\boldsymbol{C}(Q_\lambda) = \frac{1-\lambda^4}{1-\lambda^3}\boldsymbol{C}(Q). \tag{16}$$

Proof. The volume of Q_λ is $V(Q) - V(\lambda Q)$. Equating moments we have

$$(V(Q) - V(\lambda Q))\boldsymbol{C}(Q_\lambda) + V(\lambda Q)\boldsymbol{C}(\lambda Q) = V(Q)\boldsymbol{C}(Q). \tag{17}$$

But $V(\lambda Q) = \lambda^3 V(Q)$, $V(Q) - V(\lambda Q) = V(Q)(1-\lambda^3)$, and $\boldsymbol{C}(\lambda Q) = \lambda\boldsymbol{C}(Q)$, so (17) becomes $(1-\lambda^3)\boldsymbol{C}(Q_\lambda)V(Q) = (1-\lambda^4)\boldsymbol{C}(Q)V(Q)$, which implies (16). ∎

Note that Theorem 7 can be obtained as a limiting case of (16) as $\lambda \to 1$ because the shell Q_λ has constant thickness and we can regard the area centroid of the boundary surface of Q as the limiting case of the volume centroid of the shell Q_λ as $\lambda \to 1$.

There is also an extension of Theorem 11 relating the volume centroid of the shell Q_λ to the area centroid of its full boundary S_λ (inner plus outer):

Theorem 12. *The area centroid $\boldsymbol{C}(S_\lambda)$ of the full boundary of a circumgonal shell Q_λ is related to its volume centroid $\boldsymbol{C}(Q_\lambda)$ by the equation*

$$\boldsymbol{C}(S_\lambda) = \frac{4}{3}\frac{1-\lambda^3}{1-\lambda^4}\frac{1+\lambda^3}{1+\lambda^2}\boldsymbol{C}(Q_\lambda).$$

This extends Theorem 9 in [**1**], which is the planar version. We omit the proof.

Extensions to *n*-space. The main results of this paper can be extended to n-space, but we shall not pursue the details here. As might be expected, for an appropriately defined circumsolid in n-space equation (1) takes the form

$$V_n = \frac{r}{n} S_{n-1},$$

so the ratio of n-dimensional volume V_n to $(n-1)$-dimensional outer surface area S_{n-1} is r/n, where r is the inradius of the inscribed n-sphere. We also note the following n-dimensional version of Theorem 7 relating centroids of circumsolids in n-space:

$$\boldsymbol{C}(S_{n-1}) = \frac{n+1}{n} \boldsymbol{C}(V_n).$$

REFERENCES

1. T. M. Apostol and M. A. Mnatsakanian, Figures circumscribing circles, this MONTHLY **111** (2004) 853–863.
2. ———, A fresh look at the method of Archimedes, this MONTHLY **111** (2004) 496–508.
3. A. Todd, Bisecting a triangle, *ΠME Journal* **11** (1999) 31–37.

TOM M. APOSTOL joined the Caltech mathematics faculty in 1950 and became professor emeritus in 1992. He is director of *Project MATHEMATICS!* (http://www.projectmathematics.com) an award-winning series of videos he initiated in 1987. His long career in mathematics is described in the September 1997 issue of *The College Mathematics Journal*. Currently he is working with colleague Mamikon Mnatsakanian to produce materials demonstrating Mamikon's innovative and exciting approach to mathematics.
California Institute of Technology, 253-37 Caltech, Pasadena, CA 91125.
apostol@caltech.edu

MAMIKON A. MNATSAKANIAN received a Ph. D. in physics in 1969 from Yerevan University, where he became professor of astrophysics. As an undergraduate he began developing innovative geometric methods for solving many calculus problems by a dynamic and visual approach that makes no use of formulas. Currently he is working with Tom Apostol under the auspices of *Project MATHEMATICS!* to present his methods in a multimedia format.
California Institute of Technology, 253-37 Caltech, Pasadena, CA 91125.
mamikon@caltech.edu