SOLUTION OF RADIATIVE TRANSFER PROBLEM IN A PLANE LAYER FOR THE MODEL OF COMPLETE FREQUENCY REDISTRIBUTION. II. THREE-DIMENSIONAL MEDIUM

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In the first part of the study approximate analytic solutions were obtained for the problem of radiative transfer in the model of complete frequency redistribution in a one-dimensional medium of finite thickness. In the present paper analogous solutions are given for the case of a plane-parallel homogeneous layer for a spherical phase function (isotropic scattering). The accuracy of these solutions is higher than in the one-dimensional case and is a fraction of a percent at the line center. With increasing thickness the accuracy of the solutions rapidly increases. As an illustration, tables of Ambartsumyan's φ and ψ functions are given in the case of a Lorentzian profile. The asymptotic approximation for large layer thicknesses and in the neighborhood of the line center that can be extracted from the high-accuracy solutions is discussed.

1. Introduction

It appears to be in principle impossible to find analytic solutions to the problem of radiative transfer in a plane-parallel medium of finite thickness for the model of complete frequency redistribution within the line. Only asymptotic solutions are known, and these only for the X and Y functions for a layer of large optical thickness, and they are valid only in the immediate vicinity of the line center. Moreover, if these solutions are written down for the general formulation of the problem their error is too large (and increases rapidly with increasing distance from the line center), and such asymptotic solutions in the model of frequency redistribution are clearly unsatisfactory for applied purposes.

In our previous [1] we found accurate analytic solutions to the problem of radiative transfer in the case of a one-dimensional medium of finite thickness in the general model of complete redistribution. The error of these solutions is of the order of a few percent in practically the complete frequency region and already for a layer of zero thickness (at the central frequency). With increasing thickness of the layer the accuracy of these solutions rapidly increases.

In the case of a three-dimensional medium (with isotropic scattering) no fundamental difficulties arise in the finding of analogous solutions, and the corresponding final expressions can even be given formally, by analogy with the one-dimensional case. Moreover, simple considerations concerning the asymptotic nature of the approximation makes it clear a priori that in the case of a three-dimensional medium our solutions must be significantly more accurate. Naturally, there are come features that distinguish the three-dimensional case, and we begin our exposition with a discussion of them.

In the considered problem, the elementary scattering event is characterized by a frequenty redistribution function $\alpha(x)/\alpha_0$, where α_0 is the normalization constant of the function $\alpha(x)$, and by a spherical phase function (isotropic scattering). It is well known (see, for example, [2]) that in this case all quantities (without exception) that describe the scattering process in a plane-parallel medium can be expressed in their

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dependence on the frequency x (the dimensionless frequency relative to the line center) and the direction η (the cosine of the angle formed with the outer normal to one boundary) can be expressed by a single combination of these two variables, namely, $z = \eta/\alpha(x)$. As a result, in all the integral relations the procedure of twofold integration over x and η is replaced by a single integration with respect to a new variable,

$$\int_{-\infty}^{\infty}\int_{0}^{1} \ldots\, dx\,d\eta \to \int_{0}^{\infty} \ldots\, G(z)\,dz,$$

and this results in the appearance of an additional "characteristic" function G(z), which is determined by the expression

$$G(z) = \begin{cases} \dfrac{2}{\alpha_0}\displaystyle\int_0^{\infty} \alpha^2(x)\,dx, & |z| \leqslant 1, \\ \dfrac{2}{\alpha_0}\displaystyle\int_{x(z)}^{\infty} \alpha^2(x)\,dx, & |z| > 1, \end{cases} \tag{1}$$

where $\alpha(x(z)) = 1/z$, $x(z) \geq 0$.

In such a formulation the probability interpretation of the transfer process remains valid if one notes that $\frac{1}{2}\lambda G(z)$ is the probability of an absorbed photon's being re-emitted in intervals of frequencies x and directions η such that the corresponding ratio $\eta/\alpha(x)$ will lie in unit interval of the new variable z.

For example, we use the Ambartsumyan function $\frac{\lambda}{2}\varphi(z)$ for a semi-infinite medium, and this will have the meaning of the density of probability of escape of a photon from the semi-infinite medium in unit interval of the variable in the neighborhood of its given value if on the boundary of the medium there is an absorbed photon (irrespective of its initial frequency and direction). This probability differs from the usually adopted form H(z), which corresponds to a "reduced" and not ordinary probability, by the factor $\frac{\lambda}{2}G(z)$. In all that follows we shall use only ordinary probabilistic quantities; in particular, instead of the X and Y functions we will use the corresponding probabilities, since it is only the probabilistic interpretation that allows the only correct physical interpretation of the basic relations which underlie our conclusions [3]; in addition, it is perspicuous.

2. Characteristics of the Semi-Infinite Medium

For the surface Green's function of the semi-infinite medium, $\overline{Y}(\tau, x, x', \eta, \xi)$ and $\overline{Z}(\tau, x, x', \eta, \xi)$, in the new variables z and z' we introduce the following notation: $Y(\tau, z, z')$ is the density of the probability that a photon traveling at depth τ of the semi-infinite medium in the direction of its boundary with value of the variable z' escapes from the medium with value z, and $Z(\tau, z, z')$ is the analogous quantity in the case of a photon that initially is traveling in the direction into the medium. These quantities satisfy [4]

$$\begin{aligned} Y(\tau_1+\tau_2, z, z') &= \int_0^{\infty} Y(\tau_1, z, z'')\,Y(\tau_2, z'', z')\,dz'', \\ Z(\tau_1+\tau_2, z, z') &= \int_0^{\infty} Y(\tau_1, z, z'')\,Z(\tau_2, z'', z')\,dz'' \end{aligned} \tag{2}$$

and have the explicit representations [4]

$$Y(\tau, z, z') = \frac{\lambda}{2} z\varphi(z)\,\frac{F(\tau, z) - F(\tau, z')}{z - z'}\,G(z') + e^{-\tau/z}\delta(z - z'), \tag{3}$$

$$Z(\tau, z, z') = \frac{\lambda}{2} z\varphi(z) \frac{F(\tau, z) + \tilde{F}(\tau, z')}{z + z'} G(z'),$$

where the functions F and $\tilde{F}$ can be represented in the form

$$F(\tau, z) = \frac{P(\tau, z)}{\frac{\lambda}{2}\varphi(z)} = \frac{P(\tau, z)}{P(o, z)}, \quad \tilde{F}(\tau, z) = z\varphi(z)\int_0^\infty \frac{P(\tau, z')}{z + z'} G(z')\,dz', \tag{4}$$

and $P(\tau, z)$ is the density of the probability of escape of a photon absorbed at depth τ of the semi-infinite medium.

As in the case of isotropic scattering of monochromatic radiation in a three-dimensional medium we have relations of the type

$$P(\tau, \tau_0, z) = Y(\tau, \tau_0, z, 0) = Z(\tau, \tau_0, z, 0),$$

which are related to the fact that the value $z' = 0$ corresponds to a photon traveling parallel to the boundary of the medium ($\zeta = 0$), which sooner or later must be absorbed at the same depth τ in the layer τ_0 (as usual, we measure the optical depth or thickness from the central frequency of the line). Similarly, by replacing z by $-z$ (reversing the sign), we can go over, say, from Y to Z, since a change in the sign of z is equivalent to reversal of the spatial direction. Here, we shall not write out the numerous helpful relations of this kind, referring instead the reader to [4].

3. Solution of the Problem

We write down the basic relations of the reduction method, which establish a linear connection between the solutions to the transfer problem for a layer of finite thickness to those for a semi-infinite medium:

$$J^+(z) = j^+(z) + \int_0^\infty Z(\tau_0, z, z') j^-(z')\,dz', \quad J^-(z) = j^-(z) + \int_0^\infty Z(\tau_0, z, z') j^+(z')\,dz'. \tag{5}$$

Here, $j^{\pm}(z)$ is the density of the probability of escape of a photon through some particular depth of the layer of thickness τ_0 for an arbitrary distribution in it of propagating and absorbed primary photons, and $J^{\pm}(z)$ is the analogous quantity for the semi-infinite medium (see [1]).

In many respects, it is convenient to represent the solution of the problem with respect to the sum $s = j^+ + j^-$ and difference $h = j^- - j^+$ of the required quantities j^+ and j^- by addition and subtraction of Eqs. (5).

We omit the derivation of the required high-frequency analytic solutions, which is completely analogous to the derivation in the one-dimensional case [1], but we do discuss in somewhat more detail the nature of the basic approximation that is used in doing this. This approximation reduces to the replacement everywhere under the integral sign of $\tilde{F}(\tau_0, z)$ at value τ_0 corresponding to the layer thickness by its asymptotic behavior $C(\tau_0)z$. In other words, we use the approximation

$$z\int_0^\infty \ldots \tilde{F}(\tau_0, z') \frac{dz'}{z \pm z'} \to \tilde{F}(\tau_0, z)\int_0^\infty \ldots \frac{z'dz'}{z \pm z'}. \tag{6}$$

Figure 1 gives an idea of the extent to which $\tilde{F}(\tau_0, z)$ is better approximated by the asymptotic behavior $C(\tau_0)z$ than is $F(\tau_0, z)$ (tables of these functions are given in the Appendix), which has the same behavior when $\tau_0 \gg z$. At infinity both quantities tend to constant values ($\lambda \neq 1$), whereas the asymptotic behavior increases linearly with z. However, by virtue of the rapid decrease with increasing z of the other quantities that occur in the integrands of the basic approximation, such a discrepancy introduces only a slight error in the value of these integrals, and as a result our solutions are extremely accurate. (In the figure the ordinate scales are chosen in such a way that for different τ_0 the constant $C(\tau_0)$ is the same.)

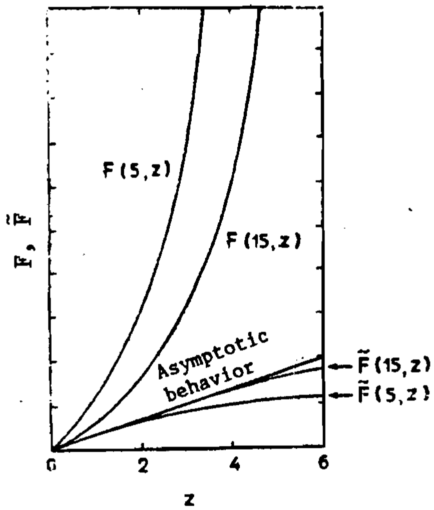

Fig. 1. Comparison of the behavior of the curves of $F(\tau, z)$ and $\tilde{F}(\tau, z)$ and their asymptotic behavior at $\tau_0 \gg z$ (Lorentzian profile).

We give the final expressions

$$\begin{cases} s(z) = S(z) - \alpha(\tau_0, z)\, s_0 - \beta(\tau_0, z)\, s_z, \\ s_z = S_z - P_z(\tau_0)\, s_0, \quad s_0 = \dfrac{S_0 - S_\beta}{1 + \alpha_0(\tau_0) - P_\beta(\tau_0)}; \end{cases} \tag{7}$$

$$\begin{cases} h(z) = H(z) + \alpha(\tau_0, z)\, h_0 + \beta(\tau_0, z)\, h_z, \\ h_z = H_z + P_z(\tau_0)\, h_0, \quad h_0 = \dfrac{H_0 + H_\beta}{1 - \alpha_0(\tau_0) - P_\beta(\tau_0)}, \end{cases}$$

where

$$\alpha(\tau_0, z) = \frac{\lambda}{2}\varphi(z)\tilde{F}(\tau_0, z), \qquad \beta(\tau_0, z) = P(\tau_0, z) - \alpha(\tau_0, z) \tag{8}$$

and we have introduced the notation

$$f_0 \equiv \int_0^\infty f(z)\, G(z)\, dz, \quad \tilde{f}_0 \equiv \int_0^\infty F(\tau_0, z) f(z)\, G(z)\, dz,$$

$$f_z \equiv z\int_0^\infty \frac{f(z')\, G(z')}{z + z'}\, dz', \quad f_\beta \equiv \int_0^\infty \beta(\tau_0, z) f_z\, G(z)\, dz = \beta_0(\tau_0) f_0 - \sqrt{1-\lambda}\,\tilde{f}_0. \tag{9}$$

These solutions have a much simpler form than the analogous solutions for the case of the monochromatic solution for the simple reason that in the case of frequency redistribution there is no characteristic constant k (in the asymptotic limit $\tau_0 \gg z$) corresponding to the discrete value of the spectrum (corresponding to an eigensolution); at the same time, the solution is not conservative ($\lambda \neq 1$). In fact, the case $\lambda = 1$ will be the subject of a separate paper.

We now turn to the derivation of approximate and asymptotic forms from our highly accurate analytic solutions; these use the approximation (6).

If we also use the approximation in which $F(\tau_0, z)$ in the integrand is replaced by its asymptotic value $C(\tau_0)z$ (but not in explicit form), then the solutions simplify appreciably, since it is then necessary to set equal to zero all moments of the type f_β, $f_\beta \to 0$. Then

TABLE 1. Approximate and Exact Values of the X and Y Functions for a Lorentzian Profile, $\lambda = 0.65$.

z \ τ_0	0.1		1		5		10		∞
	X	Y	X	Y	X	Y	X	Y	$H(z)$
0.257	1.04585	0.71843	1.09515	0.05489	1.10043	0.00405	1.10070	0.00137	1.1008
	1.04655	0.71463	1.09744	0.05767	1.10063	0.00444	1.10075	0.00148	
0.51	1.05424	0.87226	1.13746	0.21686	1.15143	0.00973	1.15196	0.00303	1.1521
	1.05033	0.85246	1.13991	0.21873	1.15180	0.01031	1.15206	0.00321	
1.0	1.06448	0.96580	1.17892	0.49601	1.21430	0.03398	1.21547	0.00726	1.2158
	1.05258	0.96042	1.17896	0.49306	1.21492	0.03454	1.21565	0.00750	
2.04	1.07874	1.02902	1.20657	0.72268	1.28712	0.17268	1.29128	0.03124	1.2920
	1.05416	0.99489	1.20902	0.78347	1.28785	0.17165	1.29159	0.03130	
3.08	1.08876	1.05519	1.23244	0.92991	1.32664	0.33806	1.33521	0.09004	1.3368
	1.05522	1.01086	1.22140	0.91011	1.32719	0.33480	1.33563	0.08945	
4.83	1.10082	1.07940	1.24847	1.04439	1.36434	0.56728	1.38025	0.23340	1.3844
	1.05664	1.02314	1.23157	1.01629	1.36447	0.56076	1.38087	0.23124	
7.03	1.11146	1.09665	1.26029	1.11484	1.39036	0.75840	1.41334	0.41012	1.4216
	1.05829	1.03050	1.23849	1.08000	1.39015	0.74920	1.41430	0.40621	
10.58	1.12331	1.11316	1.27206	1.17142	1.41349	0.94020	1.44390	0.62711	1.4590
	1.06072	1.03689	1.24529	1.12999	1.41316	0.92886	1.44558	0.62163	

Table 2. Approximate and Exact Values of the X and Y Functions for a Lorentzian Profile, $\lambda = 0.99$.

z \ τ_0	0.1		1		5		10	
	X	Y	X	Y	X	Y	X	Y
0.257	1.07658	0.74043	1.16737	0.08236	1.18804	0.01189	1.19069	0.00511
	1.07836	0.73591	1.17444	0.09073	1.14449	0.01480	1.19134	0.00656
0.51	1.09368	0.90185	1.24866	0.27831	1.29772	0.02987	1.30331	0.01216
	1.08635	0.87473	1.25613	0.28500	1.30083	0.03456	1.30464	0.01475
1.0	1.11847	1.00898	1.33691	0.60456	1.44993	0.09027	1.46224	0.03118
	1.09369	0.96817	1.33557	0.59687	1.45502	0.09580	1.46479	0.03519
2.04	1.16225	1.10274	1.43090	0.96046	1.65430	0.32679	1.68751	0.10596
	1.10360	1.01533	1.40464	0.91520	1.65857	0.32547	1.69185	0.11005
3.08	1.20042	1.16030	1.48819	1.12998	1.78277	0.57383	1.84074	0.23071
	1.11226	1.03107	1.43986	1.05017	1.78168	0.56079	1.84532	0.23183
4.83	1.25569	1.23832	1.55538	1.30090	1.92414	0.91191	2.02098	0.48980
	1.12500	1.04602	1.47504	1.16960	1.91009	0.87788	2.02292	0.28280
7.03	1.31423	1.31958	1.62005	1.42836	2.04176	1.19169	2.17543	0.78393
	1.14019	1.05800	1.50736	1.24263	2.01132	1.13520	2.17152	0.76719
10.58	1.39149	1.43135	1.70440	1.55897	2.17493	1.45090	2.34757	1.11450
	1.16360	1.07325	1.55125	1.30081	2.12244	1.36739	2.33378	1.08790

APPENDIX: The Functions $F(\tau, z)$ and $\tilde{F}(\tau, z)$ for a Lorentzian Profile, $\lambda = 0.65$.

z \ τ_0	0.1		1		5		10		15	
	F	$\tilde{F}$	F	$\tilde{F}$	F	$\tilde{F}$	F	$\tilde{F}$	F	$\tilde{F}$
.073	.3111	.2878−1	.1097−1	.9606−2	.1608−2	.1537−2	.5768−3	.5634−3	.3104−3	.3054−3
.158	.5730	.5134−1	.2588−1	.1829−1	.3318−2	.3030−2	.1175−2	.1121−2	.6301−3	.6098−3
.240	.7061	.7061−1	.5506−1	.2634−1	.5167−2	.4498−2	.1805−2	.1679−2	.9636−3	.9170−3
.325	.7843	.8782−1	.1019	.3396−1	.7208−2	.5964−2	.2477−2	.2247−2	.1316−2	.1230−2
.414	.8370	.1036	.1619	.4129−1	.9521−2	.7446−2	.3207−2	.283[illegible]−2	.1695−2	.1555−2
.510	.8743	.1185	.2282	.4847−1	.1225−1	.8966−2	.4014−2	.3439−2	.2108−2	.1896−2
.613	.9033	.1329	.2987	.5562−1	.1573−1	.1054−1	.4923−2	.4082−2	.2567−2	.2259−2
.727	.9257	.1470	.3687	.6284−1	.2049−1	.1221−1	.5975−2	.4772−2	.3086−2	.2651 2
.854	.9447	.1611	.4403	.7026−1	.2751−1	.1400−1	.7233−2	.5525−2	.3688−2	.3081−2
1.000	.9599	.1754	.5080	.7800−1	.3783−1	.1595−1	.8817−2	.6360−2	.4405−2	.3563−2
1.171	.9743	.1903	.5799	.8623−1	.5401−1	.1811−1	.1098−1	.7304−2	.5290−2	.4112 2
1.376	.9862	.2059	.6481	.9513	.7758−1	.2057−1	.1427−1	.8397−2	.6447−2	.4753−2
1.632	.9976	.2227	.7193	.1049	.1134	.2342−1	.1993−1	.9695−2	.8109−2	.5522−2
1.963	1.0075	.2412	.7882	.1161	.1641	.2683−1	.3031−1	.1128−1	.1087−1	.6475−2
2.414	1.0172	.2622	.8608	.1290	.2392	.3105−1	.5122−1	.1331−1	.1656−1	.7708−2
3.078	1.0259	.2867	.9325	.1448	.3434	.3653−1	.9272−1	.1603−1	.3024−1	.9393−2
4.165	1.0346	.3167	1.0090	.1648	.4955	.4412−1	.1798	.1997−1	.6851−1	.1188−1
6.314	1.0428	.3561	1.0867	.1924	.7071	.5573−1	.3564	.2633−1	.1780	.1603−1
12.71	1.0511	.4148	1.1706	.2363	1.0158	.7712−1	.7310	.3908−1	.5134	.2475−1

$$\begin{cases} s = S - a s_0 - \beta s_z, \\ s_z = S_z - P_z s_0, \\ s_0 = \dfrac{S_0}{1 + a_0}, \end{cases} \qquad \begin{cases} h = H + a h_0 - \beta h_z, \\ h_z = H_z + P_z \cdot h_0, \\ h_0 = \dfrac{H_0}{1 - a_0}. \end{cases} \tag{10}$$

The accuracy of these solutions is only slightly less good than that of the previous ones.

A more significant simplification, though it is associated with a greater loss of accuracy, involves in addition the approximation $F(\tau_0, z) = C(\tau_0) \cdot z$ in explicit form and not only in integrands. It is then necessary to set $\beta \to 0$, and the solutions take the asymptotic form

$$s = S - a(\tau_0, z)\, s_0, \quad h = H + a(\tau_0, z)\, h_0, \quad s_0 = \frac{S_0}{1 + a_0(\tau_0)}, \quad h_0 = \frac{H_0}{1 - a_0(\tau_0)}. \tag{11}$$

At the same time, it is in general necessary to replace the characteristics of the semi-infinite media, S and H (which contain τ_0), by their asymptotic behavior at large $\tau_0 \gg z$,

$$S \to S_{ac}, \quad H \to H_{ac}, \tag{12}$$

and we then obtain the analog of the asymptotic solutions known for the X and Y functions [2].

The errors of these solutions are too large (see [5], where the asymptotic solutions (11) are compared with the exact ones); for example, the Y function differs appreciably from the accurate one already at large τ_0.

It should be noted that in the literature there are a number of approximate formulas for the source function in the model of complete frequency redistribution; however, their errors are too large (see, for example, [6]) for a comparison to be made here with our solutions. In the best case they are accurate to within a factor of 2.

4. Numerical Solutions

To illustrate the accuracy of the analytic solutions, we give the results of calculations of the φ and ψ functions for the case of a Lorentzian profile, when

$$\varphi(\tau_0, z) = \varphi(z) - \frac{2}{\lambda}\cdot\frac{\tilde{F}(\tau_0, z)}{\varphi(z)}\beta(\tau_0, z)\left(1 - \frac{\lambda}{2}\varphi_0\right) - \frac{\lambda}{2}\varphi(z)\tilde{F}(\tau_0, z)\psi_0,$$

$$\psi(\tau_0, z) = \varphi(z)\tilde{F}(\tau_0, z)\left(1 - \frac{\lambda}{2}\varphi_0\right) + \frac{2}{\lambda}\cdot\frac{\beta(\tau_0, z)}{\varphi(z)}\left(1 - \frac{\lambda}{2}\psi_0\tilde{F}(\tau_0, z)\right), \qquad (13)$$

$$\frac{\lambda}{2}\varphi_0 = \frac{s_0 + h_0}{2}, \quad \frac{\lambda}{2}\psi_0 = \frac{s_0 - h_0}{2}.$$

For comparison, in Tables 1 and 2 we give calculations of the φ and ψ functions by means of the analytic expressions (13) (first row) and numerical solution of Eqs. (11) by the discretization method (second row). The calculations were made for 20 values of $z = \tan(\pi\sigma/2)$, where σ varies uniformly from 0 to 1 (for convenience of comparison with the well-known data in the literature we have also given the X and Y functions).

We see that the accuracy of the solutions is indeed higher than in the one-dimensional case. This is readily understood, since at the same optical depth in the three-dimensional medium the photons travel on the average at angle $\zeta = 1/2$ to the normal and are therefore subject on the average to a greater number of scatterings compared with the one-dimensional medium, and it is this that ensures the better asymptotic approximation of the function $\tilde{F}$ and, incidentally, F as well.

Further, as was to be expected, the solution for φ is comparatively more accurate than for ψ. The accuracy increases with increasing thicknesses of the layer τ_0, with decreasing λ, and as z tends to zero, i.e., as the line center is approached. However, the solutions are accurate to fractions of a percent in practically the entire range of frequencies z.

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