GENERALIZED RELATIVISTIC THEORY OF GRAVITATION

II. Baryon Configurations

G. S. Saakyan and M. A. Mnatsakanyan

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In this paper, a continuation of [1], further transformations are reported of the field equations of the generalized relativistic theory of gravitation for the interiors of static spherically symmetric mass distributions.

Limiting transition to the equations of the general theory of relativity and the correspondence principle are discussed.

Conditions on the surface and at the center are considered for which the configuration can have finite central pressure. A detailed description is given of the integration process for the interior boundary-value problem. Certain important properties of gravitars, characterized by the condition $M/R \gg 1$, are derived.

Results of a numerical integration for an incompressible fluid and a real baryon gas are compared with the corresponding results predicted by other theories.

Gravitars are models of supermassive static formations with definite values of P_0/ρ_0 and anomalously large gravitational mass defects.

§1. In [1] we obtained field equations for the interiors of spherically symmetric mass distributions. We rewrite them in the following form:

$$\frac{dP}{dr} = -\frac{P+\rho}{2-r\dfrac{k'}{k}} r \times$$

$$\times \left[8\pi k e^{\lambda} P + \frac{2}{r}\frac{k'}{k} - \frac{\zeta}{2}\left(\frac{k'}{k}\right)^2 + \frac{e^{\lambda}-1}{r}\right], \qquad (1)$$

$$\frac{d\lambda}{dr} = 16\pi k r e^{\lambda}\left[\frac{\zeta P + (1-\zeta)\rho}{3-2\zeta} + \frac{P}{2-r\dfrac{k'}{k}}\right] -$$

$$-2\frac{e^{\lambda}-1}{r}\frac{1-r\dfrac{k'}{k}}{2-r\dfrac{k'}{k}} + (2-\zeta)\frac{r\left(\dfrac{k'}{k}\right)^2}{2-r\dfrac{k'}{k}}, \qquad (2)$$

$$\frac{d}{dr}\left(\frac{1}{k}\frac{dk}{dr}\right) = 8\pi k e^{\lambda} \times$$

$$\times \left\{(\rho - P) + \frac{2-r\dfrac{k'}{k}}{3-2\zeta}\left[\zeta P + (1-\zeta)\rho\right]\right\} - \frac{e^{\lambda}+1}{r}\cdot\frac{k'}{k}, \qquad (3)$$

$$P = P(\rho), \qquad (4)$$

$$\frac{d\nu}{dr} = -\frac{2}{P+\rho}\cdot\frac{dP}{dr}, \qquad (5)$$

$$\frac{dm}{dr} =$$

$$= \left\{\frac{8\pi k}{3-2\zeta}\left[(2-\zeta)\rho + 3(1-\zeta)P\right] + \frac{k'\nu'}{2k}\right\} e^{\frac{\lambda+\nu}{2}}\cdot r^2, \qquad (6)$$

$$\frac{d\mu}{dr} = 4\pi\rho e^{\lambda/2} r^2. \qquad (7)$$

Here, r is the distance from the center of the configuration, the pressure P and the density ρ of matter are related by the equation of state given by Eq. (4); $\lambda(r)$ and $\nu(r)$ describe the metric properties of space; k(r) is the gravitational scalar; $M = \int_0^R dm$ is the active gravitational mass; and $M_0 = \int_0^R d\mu$ is the proper mass of the configuration.

At the center of the star, where r = 0, we have

$$P(0) = P_0, \quad m(0) = 0, \quad \mu(0) = 0,$$

whereas on the surface, where r = R, we must satisfy conditions determined by the exterior Heckmann solution:

$$P(R) = 0, \quad R = \frac{4hBM}{\sqrt{\tau_0}\,(\tau_0^{-h} - \tau_0^{h})},$$

$$m(R) = M,$$

$$\lambda(R) = 2\ln\left[\frac{2h}{(h+1/2)\tau_0^{h} + (h-1/2)\tau_0^{-h}}\right],$$

$$\nu(R) = \frac{1}{B}\ln\tau_0,$$

$$k(R) = \tau_0^{-\beta_0/B},$$

$$k'(R) = -2\beta_0\frac{M}{R^2}e^{\lambda(R)/2}\tau_0^{-\beta_0/B-1/2}, \qquad (8)$$

where $\beta_0 = 1/2\zeta - 3$, $B = 1 + 2\beta_0$, $h = (r\zeta^2 - 10\zeta + 7)^{1/2}/2(1-2\zeta)$ and τ_0 is the value of the variable parameter τ corresponding to the surface of the configuration.

As in the general theory of relativity (GTR) the quantity m(r) does not represent the "accumulated mass," and the condition m(0) = 0 represents the absence of a "point mass" only to the extent to which $M \equiv m(R) = \int_0^R dm$. However, the term "point mass" is valid in GTR since for $r \to 0$ the quantity m(r) assumes the significance of the mass included in a sphere of radius r as soon as $e^{\lambda(r)} \to e^{\nu(r)} \to 1$, and the spacetime at the center of the configuration is infinitesimally Euclidean. As regards the "accumulated mass defect" we have $\mu(r) - m(r) \to 0$ for $r \to 0$. We will see that, in the generalized theory, $e^{\lambda(r)} \to 0$ for $r \to 0$, and the metric in the neighborhood of the center is highly non-Euclidean. In the small region surrounding the center of the configuration the "accumulated mass defect" and the proper mass are equal. By small r we mean $r \ll r_g$ and, therefore, it is clear that gravitars (this term is used to denote models of supermassive static configurations based on the generalized theory of gravitation), for which characteristically $R \ll R_g$, are associated with very large values of the gravitational mass defect.

Equations (1)–(4) form an independent system. Transforming to the variables

$$E = e^{\lambda}, \quad f = r\frac{k'}{k}, \quad s = 8\pi k(r) r^2 \tag{9}$$

we can reduce this system of equations to a system whose order is lower by unity:

$$\frac{dP}{ds} = \frac{P+\rho}{f+2}\cdot\frac{\psi}{s}, \tag{10}$$

$$\frac{dE}{ds} = \frac{2E}{s(f+2)}(sE\Phi + 1 - f - E - \psi), \tag{11}$$

$$\frac{df}{ds} = \frac{E}{s(f+2)}\left\{s[(\rho - P) + (f-2)\Phi] - f\right\}. \tag{12}$$

In these expressions

$$\psi = \frac{1}{f-2}\left(sEP + 2f - \frac{\zeta}{2}f^2 - 1 + E\right), \tag{13}$$

$$\Phi = \frac{\rho - \zeta(\rho - P)}{3 - 2\zeta}. \tag{14}$$

Before we proceed to the conditions at the center and on the surface of the configurations let us discuss the limiting transition from the equations of the generalized theory of gravitation to the corresponding equations of the general theory of relativity.

We already know [1] that this limiting transition is complete when $|\zeta|$ reaches infinity. The exterior solution for $r > r_g = 2M$ is then found to pass over into the exterior Schwarzschild solution inside the same gravitational sphere, and for $r < r_g$ the nature of the solution does not correspond to the GTR representations. Moreover, the function k(r) tends to the constant $k_0 = 1$ (when $|\zeta| \to \infty$) when $r \geq r_g$, and to the function $k(r) = (r/r_g)^{4/3}$ when $r \leq r_g$.

Assuming $f \equiv 0$ in Eqs. (10) and (11), we obtain the following set of equations:

$$\frac{dP}{ds} = -\frac{P+\rho}{4s}(sEP + E - 1),$$

$$\frac{dE}{ds} = \frac{E}{2s}(sE\rho - E + 1), \tag{15}$$

and if we substitute $s = 8\pi r^2$ and $E = (1 - 2m/r)^{-1}$ we find that these are identical with the Einstein equations $(m(R) = M)$:

$$\frac{dP}{dr} = -\frac{P+\rho}{r(r-2m)}(4\pi r^3 P + m),$$

$$\frac{dm}{dr} = 4\pi\rho r^2.$$

Since the exterior solution is used in the role of the conditions on the surface of the configuration, the latter also go over to the corresponding conditions of the GTR problem (of course, for configurations with $R > R_g = 2M$).

Consequently, the functions P(s), E(s), and $f(s) \equiv 0$ which were obtained during the limiting transition are found to coincide with solutions of the corresponding GTR problem, since they satisfy the same equations with the same conditions. This is also the case for the functions ν, μ, and m in Eqs. (5)–(7). As regards Eq. (12), it does not necessarily become an identity when $|\zeta| \to \infty$ and $f \equiv 0$, and is nonreal during the limiting transition. The reason for this can be explained readily if we write Eq. (12) symbolically in the form Z = 0 and recall that it is obtained from a variational principle using the requirement that $Z\delta\varkappa = 0$. However, the requirement Z = 0 is not valid during the transition to the theory with constant k(r) when $\delta\varkappa = \delta k = 0$.

Thus, the limiting transition in the equations of the generalized theory to the GTR equations is executed by the formal substitution $f \equiv$ and $|\zeta| \to \infty$ ($f \equiv 0$ means that, e.g., $|\zeta k'| = 0$). At the same time, equations describing the behavior of k(r) are abandoned. Similarly, in the nonstationary problem the limiting transition is executed for $|\zeta| \to \infty$ and $k(r,t) \equiv k_0$.

§2. For the integration of the system from the surface, and also for the analysis of certain other questions, it will be convenient to transform in Eqs. (9)–(14) to the new variables

$$\tilde{s} = s/M^2, \quad \tilde{P} = M^2 P, \quad \tilde{\rho} = M^2\rho, \tag{16}$$

which correspond to the following substitution: $\tilde{r} = r/m$. We have

$$\frac{d\tilde{P}}{d\tilde{s}} = \frac{\tilde{P}+\tilde{\rho}}{f+2}\cdot\frac{\psi}{\tilde{s}},$$

$$\frac{dE}{d\tilde{s}} = \frac{2E}{\tilde{s}(f+2)}(\tilde{s}E\tilde{\Phi} + 1 - f - E - \psi),$$

$$\frac{df}{d\tilde{s}} = \frac{E}{\tilde{s}(f+2)}\left\{\tilde{s}[\tilde{\rho} - \tilde{P} + (f-2)\tilde{\Phi}] - f\right\},$$

$$\tilde{P} = M^2 P(\tilde{\rho}/M^2), \tag{17}$$

where

$$\psi = \frac{1}{f-2}\left(\tilde{s}E\tilde{P} + 2f - \frac{\zeta}{2}f^2 - 1 + E\right),$$

$$\tilde{\Phi} = \frac{\tilde{\rho} - \zeta(\tilde{\rho} - \tilde{P})}{3 - 2\zeta} = M^2\Phi.$$

For the conditions on the surface of the configurations when $\tilde{s} = \tilde{S}$ we obtain from Eq. (8) the following parametric expressions (the parameter is τ_0):

$$\tilde{S} = 8\pi k(R) R^2/M = 8\pi\tau_0^{-\beta_0/B}/w^2,$$

$$\tilde{P}(\tilde{S}) = 0,$$

$$E(\tilde{S}) = \left[\frac{2h}{(h+1/2)\tau_0^h + (h-1/2)\tau_0^{-h}}\right],$$

$$f(\tilde{S}) \equiv F = -2\beta_0 w\sqrt{E(\tilde{S})/\tau_0},$$

$$w \equiv M/R = \sqrt{\tau_0}\,(\tau_0^{-h} - \tau_0^h)/4hB. \tag{18}$$

All these conditions and the surface $\tilde{S}$ itself are uniquely determined by specifying the single parameter w = M/R (the last relation in Eq. (18) is the mutually single-valued relation between w and the parameter τ_0).

Having determined the conditions given by Eqs. (18) for a fixed value of w, we specify a certain value of the mass M in the last equation in Eqs. (17) and integrate toward the center of the configuration. Since there are no conditions for Eqs. (17) at the center for $\tilde{s} = 0$ (P_0 is the value of the function $\tilde{P}(s)$ at $\tilde{s} = 0$), it would appear that we will obtain a model of the configuration with all the possible masses M for each value of w. However, the point is that (and we shall show this presently) there is only one value of M(w) for which the pressure $\tilde{P}(0)$ at the center of the configuration is finite. For any other value of M the central pressure will diverge. To show this, we must proceed to the consideration of the conditions at the center for which the central pressure P_0 is finite.

We consider only the case $\zeta < 0$ in which we are presently interested. It is clear from Eq. (10) that the necessary condition for the central pressure P(0) to be finite is that either $|f(s)| \to \infty$ for $s \to 0$ or $\psi(0) = 0$. In other words, if $[\psi(P+\rho)/(f+2)]_{s=0} \neq 0$ then the quantity $(s\,dP/ds)_{s=0} \neq 0$ and the function P(s) at the center of the star will diverge at least logarithmically. If we assume that $|f(s)| \to \infty$ for $s \to 0$ it will follow from Eq. (13) that $E(s) \approx \zeta f^2/2 < 0$ for small s since, otherwise, the pressure P(0) will be infinite. However, the function $E(s) = e^{\lambda(s)}$ cannot be negative, so that the quantity $f(0)$ must be finite. In that case, if $f(0)$ is finite the value of E(0) will also be finite, which follows from Eq. (13) for ψ and from the requirement that the central pressure must be finite. If this is so, then we must have $\psi(0) = 0$ which means that

$$2f(0) - \frac{\zeta}{2}f^2(0) - 1 + E(0) = 0. \tag{19}$$

Let us consider Eq. (11) for small values of s:

$$s\frac{dE}{ds} = \frac{2E(0)}{f(0)+2}[1 - f(0) - E(0)].$$

If the right-hand side of this equation is not zero the function E(s) will diverge at s = 0, which is inadmissible. Consequently, we must have

$$E(0)\,[1 - f(0) - E(0)] = 0. \tag{20}$$

Solving Eqs. (19) and (20), we obtain (for $\zeta = -30$)

$$E(0) = 0, \quad f(0) = \frac{1-\sqrt{1-\zeta/2}}{\zeta/2} = 1/5, \quad 1a$$

$$E(0) = 0, \quad f(0) = \frac{1+\sqrt{1-\zeta/2}}{\zeta/2} = -1/3, \quad 1b$$

$$E(0) = 1, \quad f(0) = 0, \quad 2a$$

$$E(0) = 1 - 2/\zeta = 16/15, \quad f(0) = 2/\zeta = -1/15. \quad 2b \tag{21}$$

The determination of the fields in the interior of the mass distribution in the generalized theory of gravitation is a boundary-value problem, i.e., the conditions of the problem are specified both on the surface and at the center of the configuration. The latter were found from the physical requirement that the central pressure must be finite.

The number of conditions in Eqs. (18) and (21) exceeds by unity the order of the system of equations given by Eqs. (10)–(14). However, in cases 1a and 1b the point s = 0 is a singular point for the system, and it is well known that in such cases there may be an infinite set of solutions of the boundary-value problem. One additional condition enables us to isolate from this set the unique solution. However, in cases 2a and 2b the system has no singular point and the boundary-value problem with a single extra condition cannot be solved.

Let us now consider the behavior of the solutions of Eqs. (10)–(14) in the neighborhood of s = 0 for cases 1a and 1b. Since for s = 0 both ψ and E are found to vanish, Eqs. (11) and (12) for small s can be written as

$$\frac{dE}{ds} = \frac{2(1-f_0)}{2+f_0}\cdot\frac{E}{s}, \quad \frac{df}{ds} = -\frac{f_0}{2+f_0}\frac{E}{s},$$

$$f_0 = f(0).$$

Integration of this gives the functions E(s) and $f(s)$ in the first approximation in s^l:

$$E_{(1)} = Ds, \quad l = \frac{2(1-f_0)}{2+f_0},$$

$$f_{(1)} = f_0 - \frac{f_0}{2(1-f_0)}E_{(1)}, \tag{22}$$

where D is the integration constant. The expansion is carried out in powers of s^l. The condition $(dP/ds)_{s=0} = 0$ is satisfied for the pressure P(s). Conditions (22) are used to describe the initial behavior of the functions at the center of the configuration during the numerical integration of the system of equations, and the choice of the free parameter D ensures that all the conditions in Eq. (18) are satisfied on the surface and, of course, the conditions 1a or 1b in (21) are also satisfied. With regard to condition 2a in Eq. (21), here we have no "free parameter" and the behavior of all the functions is determined by the central pressure, e.g., $f_{(1)} = ((\rho_0 - 3P_0)/3(3-2\zeta))s$.

In addition to solutions with finite central pressures there is a whole class of solutions with infinite pressure at the center but finite total mass M of the configuration.

§3. Bearing in mind the above behavior of the unknown functions at the center of the configurations, it will be useful to integrate Eqs. (10)–(14) from the center to the surface. For the condition on the surface we use the following very convenient expression:

$$E(F) = 1 + (2-\zeta)F - \frac{3}{2}(1-\zeta)F^2. \tag{23}$$

This exterior solution is most simply obtained by substituting Q = 0 in the second equation in Eqs. (31), and then integrating the resulting linear equation

$$dE/df = 2(1 + f + E + \psi)/f,$$

$$\psi = \left(2f - \frac{\zeta}{2}f^2 - 1 + E\right)/(f-2).$$

This is contained in the Heckmann solution given by Eq. (18). The Heckmann solution itself is obtained by integrating Eq. (17) with $P = \rho = 0$ and using Eq. (23) for $E(f)$.

The solution of the interior problem can be obtained as follows. Having specified the pressure P_0 at the center s = 0 and the behavior of Eq. (22), we choose a value of the constant D such that, after numerical integration on the surface s = S, where P(s) = 0, the quantities E and $F \equiv f(S)$ satisfy Eq. (23) to a given accuracy. Having obtained E, we determine τ_0, w, and k(R) from Eqs. (8) and (18). In accordance with Eq. (9) these equations then enable us to find the radius and mass of the configuration:

$$R = \sqrt{\frac{S}{8\pi k(R)}}, \quad M = wR. \tag{24}$$

Next, to determine the proper mass M_0 and the interior solutions k(r), ν(r), λ(r), and P(r), we integrate the following set of differential equations, which is equivalent to Eqs. (1)–(7):

$$\frac{dP}{dr} = \frac{P+\rho}{r}\psi,$$

$$\frac{dE}{dr} = \frac{2E}{r}(sE\Phi - E + 1 - f - \psi),$$

$$\frac{df}{dr} = \frac{E}{r}\{s[\rho - P + (f-2)\Phi] - f\},$$

$$\frac{dk}{dr} = f\frac{k}{r},$$

$$\frac{d\nu}{dr} = -2\frac{\psi}{r},$$

$$\frac{d\mu}{dr} = 4\pi\rho\sqrt{E}\,r^2, \tag{25}$$

where

$$s = 8\pi k(r) r^2,$$

$$\psi = \frac{1}{f-2}\left(sEP + 2f - \frac{\zeta}{2}f^2 - 1 + E\right),$$

$$\Phi = \frac{\rho - \zeta(\rho - P)}{3 - 2\zeta}.$$

Integration is again performed from the center where we specify that

$$P(0) = P_0, \quad E(\nu) \approx D(8\pi\gamma)^f r^{2(1-f_0)},$$

$$f(0) = f_0, \quad \mu(0) = 0$$

with the value of D found as a result of the integration of Eq. (17); for k(r) and ν(r) we use the following expressions:

$$k(r) \approx \gamma r^{f_0}, \quad \nu(0) = \text{const}. \tag{26}$$

Taking the arbitrary values γ_1 and ν_1 for γ and $\nu(0)$, we carry out a test integration toward the surface (P = = 0) and, as a result, we obtain the values k_2 and ν_2 on the surface. We then again integrate Eq. (25) with the same conditions at the center but take the following values for γ and ν:

$$\gamma = \gamma_1\frac{k(R)}{k_2}, \quad \nu(0) = \nu(R) - (\nu_2 - \nu_1), \tag{27}$$

which the above equations show are now correct in the sense that all the conditions on the surface are satisfied. We note that k(R) and ν(R) were determined earlier from Eqs. (8) and (18) simultaneously with τ_0 and w. In this way, we obtain the interior solutions P(r), ρ(r), E(r), f(r), k(r) and ν(r), and the integral parameters R, M, and M_0 for a model with given central pressure P_0. We note that during the numerical integration it is convenient to work with the variables ln s, ln r, and ln E = λ).

§4. We must now derive certain important properties of gravitars, i.e., models characterized by large values of the parameter w = M/R. Analysis of conditions (18) and of Eq. (17), and the numerical calculations show that E ≪ 1 both on the surface and in the interior of the model, and in this approximation Eqs. (17) can be rewritten as

$$\frac{dP}{ds} = \frac{P+\rho}{f+2}\cdot\frac{\psi}{s},$$

$$\frac{dE}{ds} = \frac{2E}{s(f+2)}(sE\Phi + 1 - f - \psi),$$

$$\frac{df}{ds} = \frac{E}{s(f+2)}\{s[\rho - P + (f-2)\Phi] - f\}, \tag{28}$$

where

$$\psi = \frac{1}{f-2}\left(sEP + 2f - \frac{\zeta}{2}f^2 - 1\right),$$

$$\Phi = \frac{\rho - \zeta(\rho - P)}{3 - 2\zeta}.$$

The above equations now satisfy the following similarity transformation:

$$s \to \alpha s, \quad E \to \alpha^{-1}E, \quad f \to f, \quad P \to P, \quad \rho \to \rho. \tag{29}$$

This, in turn, means that (for a given equation of state) all the gravitars are similar models and, more important, that they all have identical central pressures. Using Eqs. (9), (18) and (24), we can write the dependence of the integral parameters of the gravitars on w in the form:

$$
\begin{aligned}
M &\approx w^{\frac{3hB-3\beta_0-1}{2B}j} \approx w^{22.4},\\
R &\approx w^{\frac{hB-\beta_0}{2B}j} \approx w^{21.4},\\
k(R) &\approx w^{\beta_0 j/B} \approx w^{-1.328},\\
e^{\lambda(R)} &\approx w^{-2hj} \approx w^{-83},\\
e^{\nu(R)} &\approx w^{-j/B} \approx w^{-83.7},
\end{aligned}
\tag{30}
$$

where $j = 1/(h - 1/2)$, $\zeta = -30$.

Of course, all the above properties are valid only asymptotically (for $w \gg 1$), starting from $w \gtrsim 0.5$.

§5. For the ultrarelativistic equation of state $\rho = aP$ the mass M cancels out from Eqs. (17) and (18), and it is readily shown that this is connected with the fact that the model in which $\rho = aP$ is satisfied throughout cannot have a finite radius. This equation of state is used for the central very dense regions of the model. Substituting

$$Q = sP$$

we can reduce the order of Eq. (17):

$$
\begin{aligned}
\frac{dQ}{\partial f} &= \frac{Q}{EQ}\left[(a+1)\psi + f + 2\right],\\
\frac{dE}{df} &= \frac{2}{G}\left[EQb_1 - f + 1 - E - \psi\right],
\end{aligned}
\tag{31}
$$

where

$$\psi = \frac{1}{f-2}\left(EQ + 2f - \frac{\zeta}{2}f^2 - 1 + E\right),$$

$$G = b_2 Q + f(b_1 Q - 1),$$

$$b_1 = \frac{a - \zeta(a-1)}{4 - 2\zeta}, \quad b_2 = a - 1 - 2b_1.$$

The following is a special solution of the system corresponding to the pressure $P(s) = Q_0/s$ which becomes infinite at the center:

$$
\begin{aligned}
Q \equiv Q_0 &= \frac{3 - 2\zeta}{3 + a(3+a) - \frac{\zeta}{2}\left[1 + a(6+a)\right]},\\
E &= d_0 + d_1 f + d_2 f^2,\\
d_0 &= \frac{a+5}{(a+1)(Q_0+1)}, \quad d_1 = -\frac{2}{Q_0+1},\\
d_2 &= \frac{\frac{\zeta}{2}(a+1) - 1}{(a+1)(Q_0+1)}.
\end{aligned}
\tag{32}
$$

When $|\zeta| \to \infty$ and $f \equiv 0$ Eqs. (31) and (32) become identical with the corresponding expressions in GTR [3].

The results of numerical calculations for a model consisting of an incompressible fluid are given below. This is determined by the two parameters P_0 and $\rho(r) \equiv \rho_0 = \text{const}$. The problem can be reduced to a one-parameter problem by substituting

$$p = P/\rho_0, \quad u = 8\pi\rho_0 k(r) r^2. \tag{33}$$

Equations (17) then assume the form

$$
\begin{aligned}
\frac{dp}{du} &= \frac{p+1}{f+2}\frac{\psi}{u},\\
\frac{dE}{du} &= \frac{2E}{u(f+2)}\left(uE\varphi + 1 - f - E - \psi\right),\\
\frac{df}{du} &= \frac{E}{u(f+2)}\left\{u\left[1 - p + (f-2)\varphi\right] - f\right\},
\end{aligned}
\tag{34}
$$

where

$$\psi = \frac{1}{f-2}\left(uEp + 2f - \frac{\zeta}{2}f^2 - 1 + E\right),$$

$$\varphi = \frac{1 - \zeta(1-p)}{3 - 2\zeta}.$$

Transition to the equations of the Einstein theory gives

$$\frac{dp}{du} = -\frac{p+1}{4u}(uEp + E - 1),$$

$$\frac{dE}{du} = \frac{E}{2u}(uE - E + 1).$$

The last equation does not contain p and the system is readily integrated for the unknown functions E^{-1} and p.

The mass and radius of the model consisting of an incompressible fluid are, respectively, given by

$$M = \sqrt{\frac{3}{4\pi\rho_0}}\, w^{3/2}, \qquad R = \sqrt{\frac{3}{4\pi\rho_0}}\, w^{1/2}. \tag{35}$$

According to GTR

$$w = \frac{1}{2}\left[1 - \left(\frac{p_0+1}{3p_0+1}\right)^2\right]. \tag{36}$$

In the Newtonian theory we have Eq. (35) and

$$w = 2p_0. \tag{37}$$

In the nonrelativistic approximation to the generalized theory the incompressible fluid model is discussed in [2], where a graph is given of the function $p_0(w)$ together with the same expressions as given by Eq. (35) for the mass and radius. There are no analytic expressions for M(w), R(w), and $w(p_0)$ in the relativistic generalized theory, except of asymptotic ones (30).

§6. The numerical integration was carried out by the Runge-Kutta method with a relative accuracy of 0.1%, using the "Nairi" electronic computer.

Figure 1 shows the exterior solution E(F). It is actually the parabola defined by Eq. (23). The four heavy points on the (E,f) plane correspond to the initial points of integration for Eq. (21). We are considering only the conditions at the center, defined by 1a in Eq. (21). Curves 1–7 represent the interior solutions (matched to the exterior solution) for models

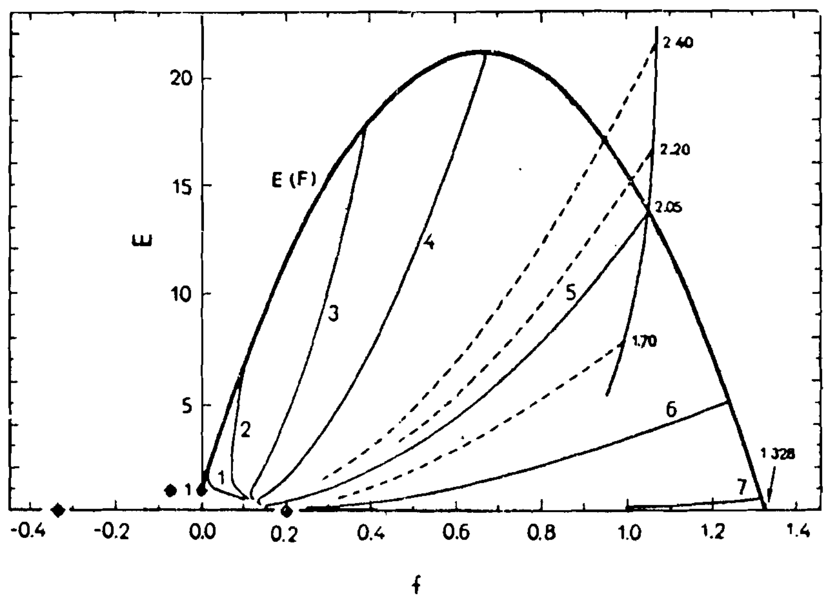

Fig. 1. Exterior solution and interior solutions for incompressible fluid models.

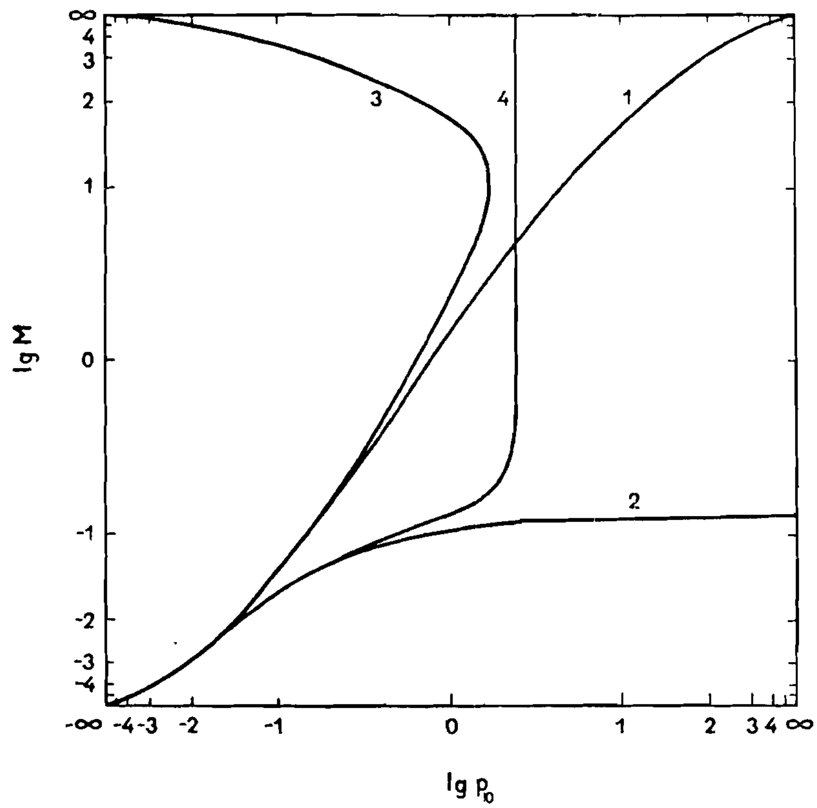

Fig. 2. Dependence of the mass M on the parameter $p_0 = P_0/\rho_0$ for the incompressible-fluid model: 1) Newtonian theory; 2) Einstein theory; 3) nonrelativistic generalized theory of gravitation; 4) relativistic generalized theory of gravitation. The scale is constructed as follows. We take the side of the square as equal to two units of length, and represent X by the quantity $2^{-|X|}$ measured from $(-\infty)$ if $X \leq 0$ and from $(+\infty)$ if $X \geq 0$. This ensures that the functions and their derivatives remain continuous. The scale is convenient because it covers the entire range of variation of the variable, $[-\infty, +\infty]$, and ensures that the central region appears magnified.

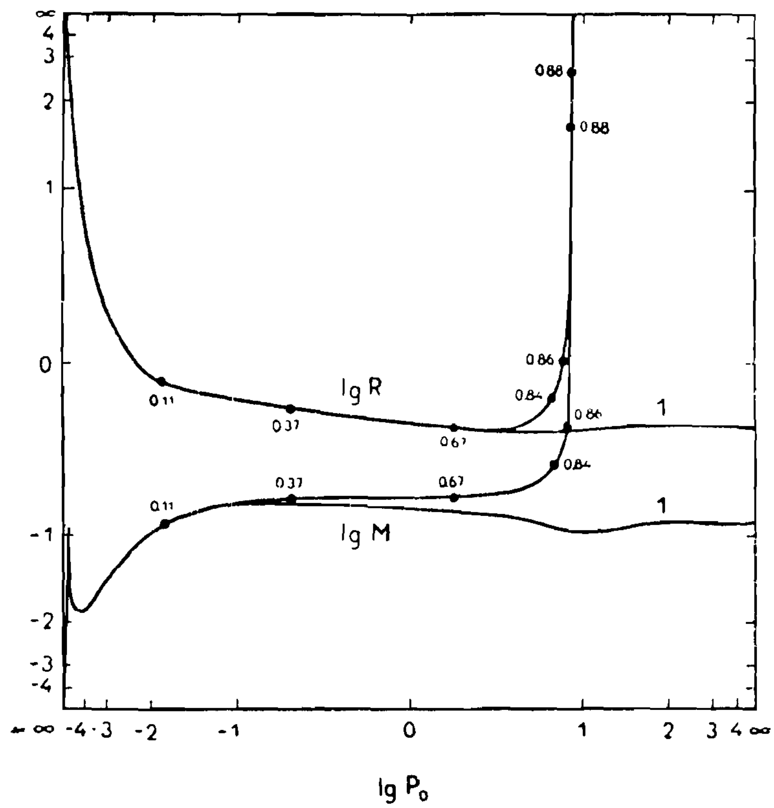

Fig. 3. Mass and radius as functions of central pressure for static spherical configurations consisting of a real baryon gas. The equation of state is taken from [9]. According to the Einstein theory (curve 1) the mass of the configurations is of the order of the solar mass and the central pressure can have any value. According to the generalized theory of gravitation (curve 2) the pressure P_0 has an upper bound but the mass can have any value with $P_0 \to 7.52$ as $M \to \infty$. Numbers marked against some of the points represent the values of $q_0 = P_0/\rho_0$.

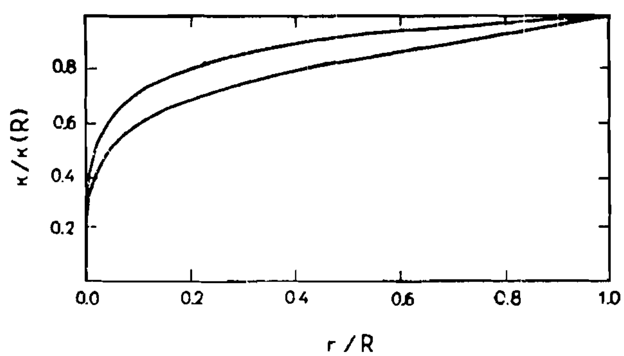

Fig. 4. The gravitational scalar as a function of distance from the center of the configuration. Curve 2 corresponds to gravitars, $q_0 = 0.88$. When we pass to models with small q_0 the function k(r) lies still higher, tending to the limit k(r) = 1 as $q_0 \to 0$.

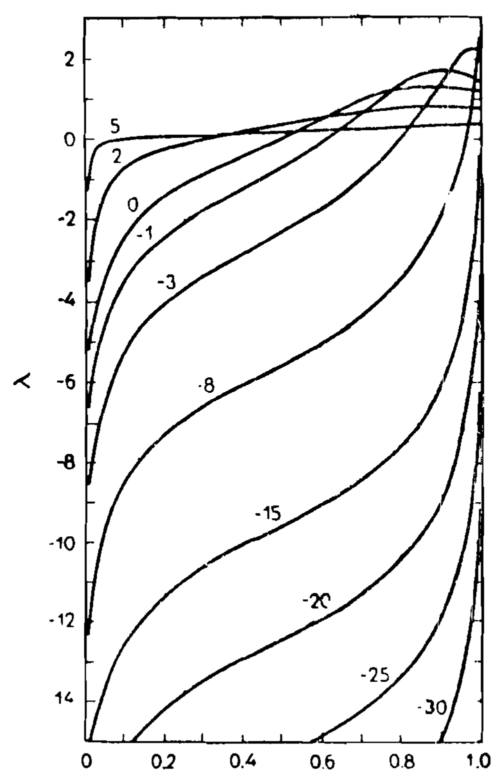

Fig. 5. Dependence of $\lambda = \ln|g_{rr}|$ on r/R. The curves lie increasingly lower as the mass of the configuration increases. For gravitars the $\lambda(r/R)$ lines are parallel to each other. Numbers marked against the curves represent values of $\lambda \equiv \ln D$.

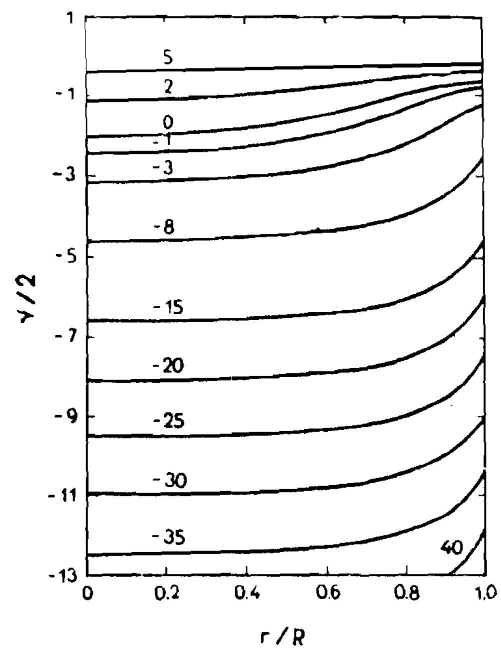

Fig. 6. Dependence of $\nu = \ln g_{00}$ on r/R. As the mass of the configuration increases, the ν(r/R) lines are shifted downward. For gravitars they are parallel to each other. Numbers marked on the curves represent the values of λ_0.

Table 1

The Most Important Parameters of Incompressible Fluid Models

ϖ	ρ_0	M	R	$k(R)$	$\lambda(R)$
0.729	2.070	$5.23\cdot10^6$	$7.16\cdot10^5$	0.612	−21.86
.657	"	$5.71\cdot10^3$	$8.70\cdot10^3$	.703	−13.18
.635	"	$1.26\cdot10^3$	$1.99\cdot10^3$	.736	−10.29
.613	"	$2.81\cdot10^2$	$4.59\cdot10^2$	.771	− 7.389
.592	"	62.46	$1.05\cdot10^2$	.808	− 4.495
.572	2.070	13.86	24.26	.846	− 1.605
.567	2.069	9.700	17.12	.856	− 1.179
.564	2.069	7.595	13.47	.862	− 0.924
.558	2.068	4.851	8.702	.874	0.382
.552	2.067	3.078	5.581	.886	1.206
.548	2.065	2.426	4.427	.893	1.618
.538	2.047	1.217	2.262	.912	2.615
.526	1.994	0.617	1.174	.931	3.045
.511	1.886	0.391	0.764	.944	2.876
.498	1.761	0.298	0.598	.952	2.617
.480	1.580	0.232	0.484	.960	2.301
.456	1.323	0.185	0.406	.966	1.960
.425	1.030	0.151	0.356	.972	1.637
.388	0.752	0.125	0.322	.977	1.346
.345	0.520	0.102	0.295	.982	1.078
.301	0.365	$8.18\cdot10^{-2}$	0.271	.986	0.861
.255	0.254	$6.32\cdot10^{-2}$	0.248	.989	0.672
.210	0.178	$4.71\cdot10^{-2}$	0.224	.992	0.518
.135	0.092	$2.42\cdot10^{-2}$	0.179	.995	0.303
.081	0.048	$1.12\cdot10^{-2}$	0.129	.997	0.172
.047	0.026	$5.01\cdot10^{-3}$	0.106	.999	0.096

with $p_0 = 0.253$, 1.323, 1.886, 1.994, 2.047, 2.065, and 2.070. The dashed lines near curves 5 represent test integrations. (We fixed the value of D in Eq. (22) and sought the corresponding values of p_0. The test solutions refer to $D = 10^{-3}$ and $p_0 = 2.4$, 2.2, and 1.7.)

Table 2

Dependence of the Mass Defect on the Parameter w in the Incompressible Fluid Model

w	0.46	0.51	0.53	0.54	0.55	0.61	0.73
ΔM	0.17	1.04	2.84	11.8	77.0	$5.9 \cdot 10^5$	$1.6 \cdot 10^{12}$

The dependence of the total mass M of the configurations on the values of p_0 is shown in Fig. 2 (curve 4). For comparison, the same figure shows the function $M(p_0)$ obtained from the Newtonian theory (curve 1), the general theory of relativity (curve 2), and the nonrelativistic variant of the generalized theory (curve 3). In the usual Newtonian theory the increase in mass leads to an unavoidable increase in central pressure. When relativistic effects are taken into account this restricts the total mass: as $p_0 \to \infty$ the mass of the incompressible fluid model reaches the maximum value of $4/9(3\pi)^{1/2}$.

In the nonrelativistic generalized theory the pressure increases with increasing mass up to a certain maximum value (p_{max}) beyond which further increase of mass leads to a reduction in pressure down to zero. For each value of the central pressure $p_0 < p_{max}$ there are two static configurations with essentially different integral parameters, M, R, and w. There are no equilibrium configurations with $p_0 > p_{max}$ for the following reasons. As the mass increases the reduced gravitational interaction begins to be felt, and to maintain the hydrostatic equilibrium it is no longer necessary for the central pressure to rise. Further increase of mass for $M > M(p_{max})$ ensures that the reduction in the gravitational interaction becomes so effective that the central pressure p_0 may then begin to fall.

In the relativistic variant of the generalized theory, the curvature of space becomes important and this leads to the saturation of the central pressure p_0 and to an almost vertical rise in the mass M for $p_0 = P_0/\rho_0 = 2.07$ at the center of the configuration. This part of curve 4 corresponds to the gravitars ($w \geqslant 0.5$) mentioned above.

Table 1 shows the most important parameters of the ideal fluid models as calculated from the relativistic generalized theory of gravitation. We note the large values of the gravitational mass defect $\Delta M = M_0 - M$ for gravitars (Table 2).

The over-all behavior of the functions k(r), E(r), and $\nu(r)$ is as follows. At the center of all the models $k(0) = E(0) = 0$ (see Eqs. (26) and (22)). As we move away from the center the function k(r) increases monotonically, passes into the exterior solution on the surface, and tends asymptotically to $k_0 = 1$ at infinity. A substantial difference between k(r) and unity is observed inside the "gravitational sphere" $r_g = 2M$ of the configuration. The function E(r) resembles the continuation of the corresponding exterior solution. From the center of the model to the surface $\nu(r)$ increases monotonically (see Eq. (5)) and

$$\nu(R) - \nu(0) = 2\int_0^R \frac{dP}{P+\rho} = 2\ln(p_0+1)$$

for an incompressible fluid.

§7. The above method was used to calculate the parameters of configurations consisting of a real baryon gas. By baryons we mean nucleons and hyperons. The equation of state was taken from [9]. Let us recall the most characteristic properties of this equation of state. It takes into account the baryon interaction energy. In the nuclear range of densities ($\rho \approx 3.6 \cdot 10^{14}$ g/cm^3) the interaction forces between the baryons are attractive and the corresponding energy is negative (it reduces pressure). Closer approach of the particles (baryon separation of the order of $2 \cdot 10^{-14}$ cm), when the density exceeds the nuclear density, is accompanied by stronger repulsive forces and the corresponding interaction energy is positive (it tends to increase the pressure). Another property of the equation of state is that at exceedingly high densities the pressure becomes equal to the energy density (for $\rho \to \infty$, $P \to \rho$). In [10] this asymptotic behavior of the relativistic baryon gas was shown to follow from the requirement that $\lim v = c$ for $\rho \to \infty$, where v is the velocity of sound. It was obtained in [11] from other considerations.

The results of our calculations are given in Table 3 and in Figs. 3–6. Figure 3 shows the mass M and radius R of the configurations as functions of the central pressure P_0. At some points on the curves we indicate the values of $q_0 = P_0/\rho_0$. The lower curves, marked 1, represent the relations deduced from the usual relativistic theory of gravitation in [12]. We can see that static configurations with any central pressure P_0 but restricted mass and radius ($M \sim M_\odot$, $R \sim 10$ km) can exist here. As $P_0 \to \infty$ the function $M(P_0)$ approaches a value which is also close to the solar mass.

It is clear from the table and from the figures that when $q_0 \geqslant 0.5$ the results of the new variant of the theory are not very different from the old variant, and essential differences appear only for $q_0 \geqslant 0.5$. As q_0 increases (and, consequently, P_0 increases) the mass and radius increase but P_0 cannot exceed 7.52 (i.e., $4.86 \cdot 10^{37}$ dyne/cm^2). In the case of the inverse relation $P_0(M)$, the central pressure increases with increasing M, and as $M \to \infty$ it tends asymptotically to the limiting value $P_0 = 7.52$. At the same time q_0 tends to the limit of 0.88. The vertical branches of curves 2 in Fig. 3 represent gravitars.

Let us now consider the interior solutions for the functions $\nu(r)$, $\lambda(r)$, and k(r). Figure 4 shows a graph of k/k(R) as a function of r/R. The upper line represents a definite baryon configuration whose integral parameters (mass, radius, etc.) are closer to the

Table 3

Some of the More Important Parameters of Configurations Consisting of a Real Baryon Gas According to the Relativistic Generalized Theory of Gravitation

P_0	q_0	R	M	w	M_0	$N_0\cdot10^{-58}$	$\nu(0)$	$\nu(R)$	λ_0	$\lambda(R)$	γ	$k(R)$
$2.23\cdot10^{-3}$	0.038	0.930	0.050	0.054	—	—	—	− 0.124	8	0.122	—	0.998
$1.59\cdot10^{-2}$	0.11	0.827	0.122	0.148	0.136	0.143	− 0.741	− 0.346	5	0.331	2.39	0.995
0.239	0.37	0.651	0.185	0.284	0.254	0.236	− 2.19	− 0.834	2	0.793	1.60	0.987
1.64	0.67	0.500	0.180	0.366	0.342	0.251	− 3.94	− 1.27	0	1.194	1.32	0.980
2.34	0.72	0.489	0.188	0.385	—	—	—	− 1.32	− 0.5	1.313	—	0.978
3.05	0.75	0.490	0.200	0.407	0.485	0.310	− 4.65	− 1.58	− 1	1.475	1.21	0.975
4.55	0.81	0.552	0.232	0.445	—	—	—	− 1.96	− 2	1.815	—	0.969
5.73	0.84	0.610	0.290	0.474	1.22	0.658	− 6.24	− 2.47	− 3	2.211	1.08	0.961
6.93	0.86	0.972	0.491	0.510	3.73	1.89	− 7.50	− 3.56	− 5	2.855	0.946	0.945
7.39	0.87	2.20	1.17	0.533	21.5	10.7	− 9.27	− 5.27	− 8	2.881	0.772	0.920
7.48	0.87	3.91	2.12	0.543	—	—	—	− 6.12	−10	2.128	—	0.902
7.49	0.87	16.9	9.51	0.563	$1.36\cdot10^{3}$	$6.72\cdot10^{2}$	−13.4	− 9.35	−15	− 0.444	0.479	0.862
7.51	0.88	74.9	43.7	0.583	$2.63\cdot10^{4}$	$1.30\cdot10^{4}$	−16.2	−12.3	−20	− 3.33	0.341	0.823
7.52	0.88	$3.27\cdot10^{2}$	197.	0.604	$5.11\cdot10^{5}$	$2.52\cdot10^{5}$	−19.2	−15.2	−25	− 6.22	0.244	0.785
7.52	0.88	$1.42\cdot10^{3}$	890.	0.626	$9.95\cdot10^{6}$	$4.91\cdot10^{6}$	−22.1	−18.1	−30	− 9.12	0.172	0.747
7.52	0.88	$6.18\cdot10^{3}$	4010.	0.648	$1.92\cdot10^{8}$	$9.46\cdot10^{7}$	−25.0	−21.0	−35	−12.01	0.123	0.716
7.52	0.88	$2.68\cdot10^{4}$	$1.81\cdot10^{4}$	0.671	$3.71\cdot10^{9}$	$1.83\cdot10^{9}$	−27.9	−23.9	−40	− 14.91	0.087	0.684

P_0 is the central pressure, ρ_0 energy density at the center, $q_0 = P_0/\rho_0$, R is the radius, M is the mass of configuration, $w = M/R$, M_0 is the proper mass (mass without gravitation), N_0 the number of baryons, $\nu(0)$ and $\nu(R)$ are values of the function $\nu(r) = \ln g_{00}(r)$ at the center and at the surface, respectively, $\lambda(R) = \ln|g_{rr}(R)|$ (g_{00} and g_{rr} are components of the metric tensor), k(R) is the gravitational scalar on the surface; $\lambda_0 = \ln D$ and γ are constants in Eq. (22) and (26), for λ and k. The mass, radius, and pressure are measured in units of 9.29 $M_\odot$, 13.7 km, and $7.47\cdot10^{36}$ dyne/cm^2, respectively. For four configurations only Eq. (17) was integrated.

parameters of the corresponding (i.e., with equal central pressure) configuration which is calculated parameters of the configuration which is calculated from the usual relativistic theory of gravitation; the lower line represents gravitars. The latter is a limiting line in the sense that there are no lines lying to the right of it. When $q_0 \to 0$ the quantity k(R) tends to the Newtonian limit k_0 and the function k/k(R) tends to unity.

The family of curves shown in Figs. 5 and 6 represents the dependence of the functions λ and ν on r/R for a series of configurations. We see that, as the mass increases, the $\lambda(r/R)$ and $\nu(r/R)$ curves lie lower and lower. In the case of gravitars, the functions $\lambda(r/R)$ and $\nu(r/R)$ are represented by parallel lines. This result, and the fact that for all gravitars the k/k(R) curves are identical if r is measured in units of R, follow from the fact that these configurations are similar.

We have also calculated configurations consisting of an ideal neutron gas. Here there are no limiting values for P_0 and q_0, and the gravitar branch is absent. The $M(P_0)$ and $R(P_0)$ curves are identical to within $1/|\zeta|$ with the corresponding curves calculated earlier on the basis of the usual theory of gravitation [6–8]. In this case, there are no appreciable deviations from the ordinary theory and the vertical branch on the $M(P_0)$ curve is absent. The essential fact here is, of course, not the chemical composition of the baryon gas; allowance for hyperon and baryon resonances does not affect the final picture. It turns out that the parameter $q_0 = P_0/\rho_0$ plays an important part here. In the case of an ideal fluid and real baryon gas new results are obtained for $q_0 \gtrsim 0.5$, and the limiting values of q_0 for which gravitars are obtained are 2.07 and 0.88 for ideal fluid and real baryon gas, respectively. For an ideal neutron gas $q \leq 1/3$ and, therefore, the region $q_0 \gtrsim 0.5$, in which one would expect appreciable deviations and the appearance of a vertical branch, is never reached in this case.

§8. Integral characteristics of spherical static configurations, for which $w \ll 1$, correspond to the usual Einstein theory. It follows from the table that for such configurations $q_0 = P_0/\rho_0$ is also small. Relativistic effects are not appreciable. Therefore, q_0 is an important parameter characterizing simultaneously both the effect of the relativistic theory of gravitation and the effect of the generalized theory of gravitation. As the parameter q_0 increases there is an increase in w, mass, radius, and central pressure (see Table 3). There is a simultaneous weakening of the gravitational interaction, i.e., a reduction in the scalar k(r), which corresponds to the maintenance of hydrodynamic equilibrium for masses greater than and pressures P_0 smaller than in the usual Einstein theory. For masses much greater than the solar mass the reduction in the mutual attraction of masses becomes so effective that the central pressure becomes saturated and increases only very slowly with further increase of mass. For stellar models consisting of a real degenerate baryon gas this limiting value of P_0 is approximately $4.86 \cdot 10^{37}$ dyne/cm^2.

All our calculations refer to the case $\zeta = -30$. An important point is the elucidation of the problem as to what happens to our results when ζ is varied. Some qualitative information about the evolution of our results with varying ζ can be obtained without performing new calculations simply by investigating the nature of the curves in Figs. 2 and 3. These curves touch eac other over the initial section or, more precisely, they are almost identical with the corresponding curves based on the usual theory and curve upward as we move to the right toward larger P_0. Next, since as $|\zeta| \to \infty$ the results of the new and the usual theories should coincide, it becomes obvious from the above nature of the lines that as $|\zeta|$ decreases the maximum value of P_0 predicted by the nonrelativistic theory and the limiting value predicted by the relativistic theory should decrease, whereas when $|\zeta|$ increases they should increase and there should be a corresponding shortening or lengthening of the coincident parts of the lines. As $|\zeta|$ increases the anomalous branches of the $M(P_0)$ lines, which represent gravitars, should shift to the right toward larger values of P_0 (larger values of q_0) and hence the length of the coincident segments should increase. The validity of this conclusion can be readily verified in the nonrelativistic variant. In fact, other things being equal, the pressure at the center of the configuration and, consequently, its maximum value are, in this case, proportional to $\sim |\zeta|^{1/2}$ (see Eqs. (32) and (33) in [2]) which directly confirms the above conclusion.

The mean density of matter in the gravitars is of the order of the nuclear density. Large masses are not obtained at lower densities (q_0): the generalized variant of the theory of gravitation is no different from the usual variant. It follows from the foregoing discussion that the limiting value of q_0 for which large effects are obtained should depend on $|\zeta|$. For $|\zeta| < 30$ the effect should be obtainable for smaller q_0 and, in particular, possibly in the case of an ideal baryon gas also.

Since they are exceedingly compact configurations, gravitars have very large gravitational mass defects (see the last line of Table 3). At first sight, it would appear that as a result of the reduction in the gravitational scalar k(r) with increasing mass, the mass defects should also increase. However, the point is that the reduction of k(r) does not mean a weakening of the gravitational field in general and may even signify the opposite. However, the point is that as the mass of the gravitars increases there is a reduction not only in k(r) but also in the components $|g_{rr}(r)|$ and $g_{00}(r)$ of the metric tensor. Therefore, the reduction in k(r) does not lead to the approach of the space-time metric to the flat metric but, on the contrary, it is accompanied by an increased curvature of space-time. This, in fact, means an enhancement of gravitation.

All the configurations in the generalized theory are characterized by the single parameter $w = M/R$. Whereas in GTR static bodies with $w > 1/2$ are impossible in principle, here we can have models of very compact equilibrium configurations with $w \gg 1$ (gravitars).

From the standpoint of the generalized theory, the general theory of relativity is invalid for these bodies (so far this can only be stated for static bodies) for which $w \geqslant 1/2$. For less compact configurations for which $w \ll 1$ both theories give the same results (correspondence principle). This is readily understood since the Schwarzschild and Heckmann solutions, which depend only on the ratio r/M, lead to identical conditions on the surface for $w \ll 1$. Transition from configurations with small w to more compact configurations at first begin to show relativistic effects for $w \approx 0.01$ and for $w \geqslant 0.3$ effects associated with the weakening of the gravitational interaction become important.

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Erevan State University

Byurakan Astrophysical Observatory